

# GIT VERSUS BAILY-BOREL COMPACTIFICATION FOR $K3$ 'S WHICH ARE QUARTIC SURFACES OR DOUBLE COVERS OF $\mathbb{P}^1 \times \mathbb{P}^1$

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**ABSTRACT.** Looijenga has introduced new compactifications of locally symmetric varieties that give a complete understanding of the period map from the GIT moduli space of plane sextics to the Baily-Borel compactification of the moduli space polarized  $K3$ 's of degree 2, and also of the period map of cubic fourfolds. On the other hand, the period map of the GIT moduli space of quartic surfaces is significantly more subtle. In our paper [LO16] we introduced a Hassett-Keel-Looijenga program for certain locally symmetric varieties of Type IV. As a consequence, we gave a complete conjectural decomposition into a product of elementary birational modifications of the period map for the GIT moduli spaces of quartic surfaces and of double covers of  $\mathbb{P}^1 \times \mathbb{P}^1$  branched over a  $(4, 4)$  curve. The purpose of this note is to provide compelling evidence in favor of our program.

Specifically, we propose a matching between the arithmetic strata in the period space and suitable strata of the GIT moduli spaces of quartic surfaces and of double covers of  $\mathbb{P}^1 \times \mathbb{P}^1$ . We then partially verify that the proposed matching actually holds. In the case of quartic surfaces we do this “by hand” for some of the strata. In the case of double covers of quadrics, we set up a VGIT which provides a one-to-one correspondence between the conjectural Mori chamber decomposition for the period space and the chamber decomposition for the moduli space of double covers coming from variation of the polarization defining (semi)stability. We prove that the correspondence holds for “half” of the chambers, while for the other half further work is required.

## INTRODUCTION

The general context of our paper is the search for a geometrically meaningful compactification of moduli spaces of polarized  $K3$  surfaces, and similar varieties (with Hodge structure of  $K3$  type). While there exist well-known geometrically meaningful compactifications of moduli spaces of smooth curves and of (polarized) abelian varieties, the situation for  $K3$ 's is much murkier. The basic fact about the moduli space of degree- $d$  polarized  $K3$  surfaces  $\mathcal{K}_d$  is that, as a consequence of Torelli and properness of the period map, it is isomorphic to a locally symmetric variety  $\mathcal{F}_d = \Gamma_d \backslash \mathcal{D}$ , where  $\mathcal{D}$  is a 19-dimensional Type IV Hermitian symmetric domain, and  $\Gamma_d$  is an arithmetic group. As such,  $\mathcal{K}_d \cong \mathcal{F}_d$  has natural compactifications (Baily-Borel, toroidal, etc.), but the remaining question is the geometric meaning of those (by way of comparison, we recall that the second Voronoi toroidal compactification of  $\mathcal{A}_g$  is modular, cf. Alexeev [Ale02]). The most natural approach to this question is to compare compactifications of  $\mathcal{F}_d$  with more geometric ones, e.g. those given by GIT, via the period map. The most basic compactification of  $\mathcal{F}_d$  is the one introduced by Baily-Borel; we denote it by  $\mathcal{F}_d^*$ . In ground-breaking work, Looijenga [Loo03a, Loo03b] gave a framework for the comparison of GIT and Baily-Borel compactifications of moduli spaces of low degree  $K3$  surfaces and similar examples (e.g. cubic fourfolds). Roughly speaking, Looijenga proved that, under suitable hypotheses, natural GIT birational models of a moduli space of polarized  $K3$  surfaces can be obtained by arithmetic modifications from the Baily-Borel compactification. In particular, Looijenga and others have given a complete, and unexpectedly nice, picture of the period map for the GIT moduli space of plane sextics (which is birational to the moduli space

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of polarized  $K3$ 's of degree 2), see [Sha80, Loo86, Fri84], and for the GIT moduli space of cubic fourfolds (which is birational to the moduli space of polarized hyperkähler varieties of Type  $K3^{[2]}$  with a polarization of degree 6 and divisibility 2), see [Loo09, Laz09b, Laz10]. By contrast, at first glance, Looijenga's framework appears not to apply to the GIT moduli space of quartic surfaces (and their cousins, double EPW sextics): [Sha81] and [O'G16] showed that the GIT stratification of moduli spaces of quartic surfaces and EPW sextics, respectively, is much more complicated than the analogous stratification of the GIT moduli spaces of plane sextics or cubic fourfolds, and there is no decomposition of the (birational) period map to the Baily-Borel compactification into a product of elementary modifications as simple as that of the period map of degree 2  $K3$ 's or cubic fourfolds. In our paper [LO16], we refined Looijenga's work and we proved that, morally speaking, Looijenga's framework can be successfully applied to the period map of quartic surfaces and EPW sextics. In fact, we have noted that Looijenga's work should be viewed as an instance of the study of variation of (log canonical) models for moduli spaces (a concept that matured more recently, starting with the work of Thaddeus [Tha96], and continued, for example, with the so-called Hassett–Keel program). This led to the introduction, in [LO16], of a program, which might be dubbed *Hassett–Keel–Looijenga program*, whose aim is to study the log-canonical models of locally symmetric varieties of Type IV equipped with a collection of Heegner divisors (in that paper we concentrated on a specific series of locally symmetric varieties and Heegner divisors, but the program makes sense in complete generality). In particular, in [LO16] we made very specific predictions for the decomposition into products of elementary birational modifications of the period maps for the GIT moduli spaces of quartic  $K3$  surfaces and of double covers of  $\mathbb{P}^1 \times \mathbb{P}^1$  branched over  $(4, 4)$  curves.

Our predictions are in the spirit of Looijenga [Loo03b], i.e. the elementary birational modifications are dictated by arithmetic. There are two related questions arising here: First, the various strata in the period space should correspond to geometric strata in the GIT compactification. Secondly, our work in [LO16] is only predictive, i.e. there is no guarantee that the given list of birational modifications is complete, or even that all these modifications occur. The purpose of this note is to partially address these two questions. Namely, we give what we believe to be a complete matching between the geometric and arithmetic strata, thus answering the first question. We view this answer as strong evidence towards the completeness and accuracy of our predictions. Furthermore, in the case of the GIT moduli space of double covers of  $\mathbb{P}^1 \times \mathbb{P}^1$  branched over  $(4, 4)$  curves we can go further on both questions by using VGIT and ideas similar to those of [CMJL12, CMJL14]. While our previous paper [LO16] looks at the period map from the point of view of the target (the Baily-Borel compactification of the period space), the present paper's vantage point is that of the GIT moduli spaces of quartic surfaces and of double covers of  $\mathbb{P}^1 \times \mathbb{P}^1$  branched over  $(4, 4)$  curves: we get what appear to be two snapshots of one and the same decomposition of the period map into a product of simple birational modifications.

Let us discuss more concretely the content of this note, and its relationship to [LO16]. To start with, we recall that in [LO16] we have introduced, for each  $N \geq 3$ , an  $N$ -dimensional locally symmetric variety  $\mathcal{F}(N)$  associated to the  $D$  lattice  $U^2 \oplus D_{N-2}$ . The space  $\mathcal{F}(19)$  is the period space of degree-4 polarized  $K3$  surfaces,  $\mathcal{F}(18)$  is the period space for hyperelliptic degree-4  $K3$ 's, i.e. double covers of a (possibly singular) quadric surface  $Q$  branched over divisors in  $|\omega_Q^{-2}|$  with ADE singularities, and  $\mathcal{F}(20)$  is the period space of degree-2 polarized hyperkählers of Type  $K3^{[2]}$  modulo the “duality” involution (or of polarized hyperkählers of Type  $K3^{[3]}$  with a polarization of degree 4 and divisibility 2). The main goal of that paper is to predict the behavior of the schemes  $\mathcal{F}(N, \beta) = \text{Proj} R(\mathcal{F}(N), \lambda(N) + \beta \Delta(N))$  for  $\beta \in [0, 1] \cap \mathbb{Q}$ , where  $\lambda(N)$  is the Hodge (automorphic) divisor class on  $\mathcal{F}(N)$ , and  $\Delta(N)$  is a “boundary” divisor, which has a clear geometric meaning in

the three cases described above. In the present paper we will concentrate our attention on  $\mathcal{F}(19)$  and  $\mathcal{F}(18)$ .

For all  $N$ , the scheme  $\mathcal{F}(N, 0)$  is the Baily-Borel compactification  $\mathcal{F}(N)^*$ . At the other extreme, for  $N = 19, 18$ , the scheme  $\mathcal{F}(N, 1)$  is isomorphic to a natural GIT moduli space  $\mathfrak{M}(N)$  (and we are confident that the same remains true for  $N = 20$ ). Specifically,

$$\mathfrak{M}(19) := |\mathcal{O}_{\mathbb{P}^3}(4)| // \mathrm{PGL}(4)$$

is the GIT moduli space of quartic surfaces in  $\mathbb{P}^3$ , and

$$\mathfrak{M}(18) := |\mathcal{O}_{\mathbb{P}^1}(4) \boxtimes \mathcal{O}_{\mathbb{P}^1}(4)| // \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1).$$

is the GIT moduli space of  $(4, 4)$ -curves on  $\mathbb{P}^1 \times \mathbb{P}^1$ . Let  $N \in \{19, 18\}$ . The period map

$$\mathbf{p}_N : \mathfrak{M}(N) \dashrightarrow \mathcal{F}(N)^*$$

is birational by Global Torelli. We expect (following Looijenga) that the inverse  $\mathbf{p}_N^{-1}$  decomposes as the product of a  $\mathbb{Q}$ -factorialization, a series of flips, and, at the last step, a divisorial contraction.

In order to be more specific, we need to describe the boundary divisor  $\Delta(N)$  for  $N \in \{19, 18\}$ . First, let  $H_h(19), H_u(19) \subset \mathcal{F}(19)$  be the (prime) divisors parametrizing periods of hyperelliptic degree 4 polarized  $K3$ 's, and unigonal degree 4 polarized  $K3$ 's respectively, and let  $H_u(18) \subset \mathcal{F}(18)$  be the divisor parametrizing periods of double covers of a singular quadric surface. The boundary divisor is given by

$$\Delta(N) := \begin{cases} (H_h(19) + H_u(19))/2 & \text{if } N = 19, \\ H_h(18)/2 & \text{if } N = 18. \end{cases}$$

The birational transformations mentioned above are obtained by considering  $\mathcal{F}(N, \beta)$  for  $\beta \in [0, 1] \cap \mathbb{Q}$ . In order to simplify notation, from now on we let  $\mathfrak{M} := \mathfrak{M}(19)$ ,  $\mathfrak{M}_h := \mathfrak{M}(18)$ ,  $\mathcal{F} := \mathcal{F}(19)$ ,  $\mathcal{F}(\beta) := \mathcal{F}(19, \beta)$ ,  $\mathcal{F}_h := \mathcal{F}(18)$ ,  $\mathcal{F}_h(\beta) := \mathcal{F}(18, \beta)$ ,  $\Delta := \Delta(19)$ ,  $\Delta_h := \Delta(18)$ , etc.

The main result of our previous paper is the prediction of the critical values of  $\beta$  corresponding to the flips, together with the description of (the candidates for) the centers of the flips on the  $\mathcal{F}$  and  $\mathcal{F}_h$  side. In fact in [LO16] we have defined towers of closed subsets (see (1.13))

$$Z^9 \subset Z^8 \subset Z^7 \subset Z^5 \subset Z^4 \subset Z^3 \subset Z^2 \subset Z^1 = \mathrm{supp} \Delta \subset \mathcal{F},$$

and (see (1.16))

$$Z_h^8 \subset Z_h^7 \subset Z_h^6 \subset Z_h^4 \subset Z_h^3 \subset Z_h^2 \subset Z_h^1 = \mathrm{supp} \Delta_h \subset \mathcal{F}_h.$$

where  $k$  denotes the codimension ( $Z^6$  and  $Z_h^5$  are missing, *no typo*), and our prediction is that the center of the  $n$ -th flip (corresponding to the critical value  $\beta_i$ ) is the closure of the strict transform of the  $n$ -th term in the relevant tower (the  $\mathbb{Q}$ -factorialization corresponds to small  $0 < \beta$ , hence the corresponding critical  $\beta$  is  $\beta_0 = 0$ ). The last critical value of  $\beta$ , i.e.  $\beta = 1$  corresponds to the contraction of the strict transform of the boundary divisor.

On the other side, Shah [Sha81] has defined closed loci  $\mathfrak{M}^{IV} \subset \mathfrak{M}$  and  $\mathfrak{M}_h^{IV} \subset \mathfrak{M}_h$  containing the indeterminacy loci of the period maps (and, a posteriori, coinciding with the indeterminacy loci), which have natural stratifications (see **Definition 3.5**)

$$\mathfrak{M}^{IV} = (W_8 \sqcup \{v\}) \supset (W_7 \sqcup \{v\}) \supset (W_6 \sqcup \{v\}) \supset (W_4 \sqcup \{v\}) \supset (W_3 \sqcup \{v\}) \supset (W_2 \sqcup \{v\}) \supset (W_1 \sqcup \{v\}) \supset (W_0 \sqcup \{v\}),$$

(here  $v$  is the point corresponding to the tangent developable of a twisted cubic curve, and the index denote dimension) and (see **Definition 3.14**)

$$\mathfrak{M}_h^{IV} = W_{h,7} \supset W_{h,6} \supset W_{h,5} \supset W_{h,3} \supset W_{h,2} \supset W_{h,1} \supset W_{h,0}.$$

As predicted by Looijenga, and refined by us, we expect that the center in  $\mathfrak{M}$  (respectively  $\mathfrak{M}_h$ ) corresponding to the center  $Z^k$  (respectively  $Z_h^k$ ) is  $W_{k-1} \sqcup \{v\}$  (respectively  $W_{h,k-1}$ ).

The purpose of this note is to give evidence in favor of the above matching. For polarized  $K3$ 's of degree 4, we prove that the described matching holds for  $Z^1$  and  $Z^2$  (equivalently, for  $(W_1 \sqcup \{v\})$

and  $(W_0 \sqcup \{v\})$ ), and we provide evidence for the matching between  $Z^9, Z^8, Z^7$  and  $(W_8 \sqcup \{v\})$ ,  $(W_7 \sqcup \{v\})$ ,  $(W_6 \sqcup \{v\})$  respectively.

For polarized hyperelliptic  $K3$ 's of degree 4 the situation is even better. Namely, in that situation, we can match the HKL program for  $\mathcal{F}_h$  with a variation of GIT for  $(2, 4)$  complete intersection curves in  $\mathbb{P}^3$ . Thus, modulo some technical issues on the VGIT side, we get a complete proof of the validity of the predictions in [LO16] for  $N = 18$  (plus geometric meaning for them). Since the predictions of [LO16] are inductive, we see the computation for hyperelliptic  $K3$ 's of degree 4 as giving strong evidence for the case of  $K3$ 's of degree 4.

In **Section 1** we give a very brief overview of the framework developed by Looijenga in order to compare the GIT and Baily-Borel compactifications of moduli spaces of polarized  $K3$  surfaces, or similar varieties, and we will illustrate it by giving a bird's-eye-view of the period map for degree-2  $K3$ 's and cubic fourfolds. We then introduce the point of view developed in [LO16], and we describe in detail the predicted decomposition of the inverse of the period maps for quartic surfaces and double covers of  $\mathbb{P}^1 \times \mathbb{P}^1$  ramified over  $(4, 4)$  curves as products of elementary birational maps (i.e. flips or contractions), see (1.8).

We continue in **Section 2**, by revisiting the work of Shah [Sha81] on the GIT for quartic surfaces and for  $(4, 4)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$ . Usually, in a GIT analysis, by boundary one understands the locus (in the GIT quotient) parameterizing strictly semistable objects, which then can be stratified in terms of stabilizers of the polystable points (see Kirwan [Kir85]). In his works on periods of quartic surfaces, Shah (see [Sha79, Sha80]) has noted that a more refined stratification is necessary, resulting into four Types of quartic surfaces, labeled I–IV, with corresponding locally closed subsets of  $\mathcal{M}$  denoted  $\mathcal{M}^I, \dots, \mathcal{M}^{IV}$ . A quartic is of Type I–III if it is cohomologically insignificant (or from a more modern point of view, it is semi-log-canonical), and thus the period map extends over the open subset of the moduli space parametrizing such surfaces; moreover the Type determines whether the period point belongs to the period space (Type I), or it belongs to one of the Type II or Type III boundary components of the Baily-Borel compactification. The remaining surfaces are of Type IV, in particular the indeterminacy locus of  $\mathbf{p}: \mathcal{M} \dashrightarrow \mathcal{F}(19)^*$  is contained in  $\mathcal{M}^{IV}$  (and *a posteriori* it coincides with  $\mathcal{M}^{IV}$ ). While for the degree-2 case, the Type IV locus consists of a single point (corresponding to the triple conic), for quartic surfaces the Type IV locus is of big dimension and it has a complicated structure. Of course, we have an analogous partition of the moduli space  $\mathcal{M}_h$  of  $(4, 4)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$  into locally closed subsets  $\mathcal{M}_h^I, \dots, \mathcal{M}_h^{IV}$ , and similar comments apply to those subsets. In our revision of Shah's work, we shed some light on the structure of Type IV (and Type II and III) loci. While arguably everything that we do here is contained in Shah, we believe that the structure becomes transparent only after one knows the predicted arithmetic behavior. In some sense, the main point of Looijenga is to bring order to the world of GIT quotients of varieties of  $K3$  type, by relating it to the orderly world of hyperplane arrangements.

In **Section 3**, we define partitions of  $\mathcal{M}^{II}$  and  $\mathcal{M}^{III}$  into locally closed subsets (our partitions are slightly finer than partitions which have already been defined by Shah in [Sha81]), and we define the stratifications of  $\mathcal{M}^{IV}$  and  $\mathcal{M}_h^{IV}$  discussed above.

**Section 4** discusses Looijenga's  $\mathbb{Q}$ -factorialization of  $\mathcal{F}^*$ , that we denote  $\widehat{\mathcal{F}}$ , and the matching between the irreducible components of  $\mathcal{M}^{II}$  (i.e. the elements of the partition of  $\mathcal{M}^{II}$  defined in **Section 3**) and the irreducible components of  $\mathcal{F}^{II}$  (i.e. the Type II boundary components of  $\mathcal{F}^*$ ). From our point of view, Looijenga's  $\mathbb{Q}$ -factorialization of  $\mathcal{F}^*$  is nothing else but  $\mathcal{F}(\epsilon)$  for  $\epsilon > 0$  small (the prediction of [LO16] is that  $0 < \epsilon < 1/9$  will do). We compute the dimensions of the inverse images in  $\widehat{\mathcal{F}}$  of the Type II boundary components of  $\mathcal{F}^*$ . Lastly, we match the irreducible components of  $\mathcal{M}^{II}$  and the Type II boundary components of  $\mathcal{F}^*$ . This matching deserves a more detailed discussion elsewhere. On the GIT side,  $\mathcal{M}^{II}$  has 8 components (of varying dimension), while  $\mathcal{F}^*$  has 9 Type II boundary components (as computed in [Sca87]), each of them is a modular

curve. By adapting arguments of Friedman in [Fri84], we can match each of the 8 components of  $\mathfrak{M}^{II}$  to one of the 9 Type II boundary components of  $\mathcal{F}^*$ , and hence exactly one Type II boundary component is left out. The discrepancy of dimensions between GIT and Baily-Borel strata (for these 8 cases) is explained by Looijenga’s  $\mathbb{Q}$ -factorialization of the Baily-Borel compactification (one of the main results of [Loo03b]). A mystery, at least for us, was the presence of a “missing” Type II boundary of  $\mathcal{F}^*$ . This has to do with what we call the second order corrections to Looijenga’s predictions (one of the main discoveries of [LO16]). For hyperelliptic degree 4  $K3$  surfaces a similar relationship between the Baily-Borel boundary (and the associated Looijenga  $\mathbb{Q}$ -factorialization) and Type II and Type III GIT boundary components (as studied by Shah [Sha81, Theorem 4.8]) holds, but we will not discuss any details in this paper.

In **Section 5** we provide evidence in favor of the predictions of [LO16] for  $\mathbf{p}: \mathfrak{M} \dashrightarrow \mathcal{F}^*$ . We start by showing that the period map behaves as predicted in neighborhoods of the points  $v, \omega \in \mathfrak{M}$  corresponding to the tangent developable of a twisted cubic curve and a double (smooth) quadric respectively. By blowing up those points one “improves” the indeterminacy locus of the period map; the exceptional divisor over  $v$  maps regularly to the (closure) of the unigonal divisor in  $\mathcal{F}^*$ , the exceptional divisor over  $\omega$  maps to the (closure) of the hyperelliptic divisor  $H_h$  in  $\mathcal{F}^*$ , and the image of the set of regular points for the map in  $H_h$  is precisely the complement of  $\Delta^{(2)}$ . This result is essentially present in [Sha81] (and belongs to “folk” tradition); we take care in specifying the weighted blow up that one needs to perform around  $v$  in order to make the map regular above  $v$ . In the language that we introduced previously, the above results match  $Z^1$  with  $W_0 \sqcup \{v\}$ . Next, we match  $Z^2$  and  $W_1 \sqcup \{v\}$ . This is the first flip in the chain of birational modifications transforming the GIT into the Baily-Borel compactification, and it is more involved than the blow-ups of  $v$  and  $\omega$ . Here it suffices to mention that  $W_1$  parametrizes quartics  $Q_1 + Q_2$ , where  $Q_1, Q_2$  are quadrics tangent along a smooth conic. (Warning: we do not provide full details of some of the proofs.) Lastly, we provide evidence in favor of the matching of  $Z^9, Z^8, Z^7$  and  $W_8 \sqcup \{v\}, W_7 \sqcup \{v\}, W_6 \sqcup \{v\}$ . It is interesting to note that the flips of  $Z^9, Z^8, Z^7$  (and similarly for  $Z_h^8, Z_h^7, Z_h^6$ ) are associated to the so-called Dolgachev singularities (aka triangle singularities or exceptional unimodular singularities)  $E_{12}, E_{13}$ , and  $E_{14}$  respectively. These are the simplest non-log canonical singularities, essentially analogous to cusp for curves. The geometric behavior of variation of  $\mathcal{F}(\beta)$  at the corresponding critical values is analogous to the behavior of the Hassett-Keel space  $\mathfrak{M}_g(\alpha)$  around  $\alpha = \frac{9}{11}$  (when stable curves with an elliptic tail are replaced by curves with cusps, see [HH09]). While hints of this behavior exist in the literature (see Hassett [Has00], Looijenga [Loo83], Shepherd-Barron, and Gallardo [Gal13]), our  $\mathcal{F}(\beta)$  example is the first genuine analogue of a Hassett-Keel behavior for surfaces (the existence of this is well-known speculation among experts in the field).

In the final section, **Section 6**, we give very strong evidence in favour of our predictions for  $\mathbf{p}_h: \mathfrak{M}_h \dashrightarrow \mathcal{F}_h^*$ . The reason of discussing the hyperelliptic quartics here (and the new part compared to quartics) is that a hyperelliptic quartic is determined by the ramification curve, in this case a  $(2, 4)$  complete intersection in  $\mathbb{P}^3$ . The moduli and the GIT analysis for curves is much easier than for surfaces, consequently one can check much more. Specifically, following quite closely [CMJL12, CMJL14] (which treats the  $(2, 3)$  complete intersection case in connection to the Hassett-Keel program for genus 4 curves), we set up a variation of GIT for  $(2, 4)$  complete intersection curves. Namely, let

$$\mathbb{P}E = \{([f_2], [\bar{f}_4]) \mid f_i \in \Gamma(\mathcal{O}_{\mathbb{P}^3}(i)), \bar{f}_4 = f_4|_{V(f_2)}\}$$

be the natural parameter space for  $(2, 4)$  complete intersections (N.B. it differs from the Hilbert scheme along the locus where  $f_2$  and  $f_4$  share a linear factor). Since  $\mathbb{P}E$  is a projective bundle over the space of quadrics, its Picard number is 2 and we can naturally set-up a one-parameter VGIT  $\mathfrak{M}_h(t) = \mathbb{P}E //_t \mathrm{SL}(4)$  (see (6.1)) for  $t \in \mathbb{Q}_+$ . Morally speaking,  $\frac{1}{t}$  is the weight given to the quadric  $V(f_2)$  in deciding the stability of the curve  $V(f_2, f_4)$ . Thus, if  $t$  is small (specifically  $t < \frac{1}{6}$ ),



it is easy to see that  $t$ -semistable curves  $V(f_2, f_4)$  are those sitting on the smooth quadric, and the stability condition is precisely that for  $(4, 4)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$  (which is included in Shah’s paper, see [Sha80, Theorem 4.8]). For  $t > \frac{1}{6}$ , via the natural period map  $\mathfrak{M}_h(t)$  is isomorphic in codimension 1 to  $\mathcal{F}_h(\beta(t))$ , and we can establish

$$\mathcal{F}_h(\beta(t)) \cong \mathfrak{M}_h(t)$$

where  $\beta(t) = (1 - 2t)/4t$ . With this description and an analysis of VGIT on  $\mathbb{P}E$  we establish that (1)  $\mathcal{F}_h(\beta)$  are well defined (they are initially defined as Proj of rings of sections, which we have not yet checked to be finitely generated), and (2) check that the [LO16] predictions are complete for hyperelliptic degree 4  $K3$ s (see **Theorem 6.3** for the precise statement). Unfortunately, there is a small wrinkle here. Namely, according to [Ben14], the polarization  $L_t$  defining the GIT quotient  $\mathfrak{M}_h(t)$  is ample only for  $t \in (0, \frac{1}{3})$ . To completely check our predictions from [LO16], we would need GIT models  $\mathfrak{M}_h(t)$  for  $t \in [\frac{1}{6}, \frac{1}{2}]$  (or equivalently  $\beta(t) \in [0, 1]$ ). The same issue arises also in [CMJL14]. There the issue is handled by considering quotients of the Hilbert scheme of  $(2, 3)$  complete intersections, and showing that from the numerical criterion perspective, one can still work on  $\mathbb{P}E$ . We explain in **Section 6** that this might be a feasible approach also in our situation, but this remains to be investigated in future work. Nonetheless, we can still define a numerical stability (see [CMJL14, Definition 4.2]) and do VGIT on  $\mathbb{P}E$  for  $t \in (0, \frac{1}{2}]$  (for  $(0, \frac{1}{3}]$  this is well defined and has usual meaning, while for  $t > \frac{1}{3}$  it is a priori only numerical, i.e. it makes only predictions; however, based on the experience of [CMJL14] we expect to be accurate and have GIT interpretation). What we get is the striking fact that the GIT predictions (which are genuinely realized in half of the range) of **Section 6** precisely coincide with the arithmetic predictions of [LO16] (see **Theorem 6.3**). (Again, we emphasize that in the range  $t \in (0, \frac{1}{3}]$  we have complete results and proofs, while for  $t \in (\frac{1}{3}, \frac{1}{2}]$  the results are partially conjectural, but with a clear road map on how to proceed.) We view this as very strong and convincing evidence towards the validity of the [LO16] predictions. Furthermore, this fits very well with the evidence that we gave in **Section 5** for quartic surfaces (see esp. Table 6 and **Remark 6.37**).

To conclude, we believe that while further work is needed (and small adjustments might occur), there is very strong evidence that our predictions from [LO16] are accurate. In any case, Looijenga’s visionary idea that the natural (or “tautological”) birational models (such as GIT) of the moduli space of polarized  $K3$ s are controlled by the arithmetic of the period space is validated in the highly non-trivial case of quartic surfaces (by contrast, in the previous known examples [Sha80, Loo86, Loo09, Laz10, LS07, ACT11] only first order phenomena were visible, and thus a bit misleading). As possible applications of our program, starting from the period domain side, one can bring structure and order to the (a priori) wild side of GIT. Conversely, starting from GIT and the work of Kirwan [Kir85, Kir89], one can follow our factorization of the period map (and do “wall crossing” computations) and compute say the Betti numbers of the moduli of quartic  $K3$  surfaces.

**Notation/Convention 0.1.** A  $K3$  surface is a complex projective surface  $X$  with DuVal singularities, trivial dualizing sheaf  $\omega_X$ , and  $H^1(\mathcal{O}_X) = 0$ .

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## 1. GIT vs. BAILY-BOREL FOR LOOIJENGA'S FRAMEWORK OF TYPE IV

The purpose of this section is to give a very brief account of Looijenga's framework and our enhancement from [LO16] (with a focus on quartics). We start with the simplest non-trivial example that fits Looijenga's framework - the case of degree-2  $K3$  surfaces (see [Sha80], [Loo86], [Fri84, §5], and [Laz16, §1] for a concise account). We then briefly touch on the general case, and we recall how it applies to the moduli space of cubic fourfolds. Lastly, we describe in detail our (conjectural) decomposition into a product of elementary birational modifications of the period map from the GIT moduli space of quartic surfaces to the Baily-Borel compactification  $\mathcal{F}^*$ , see [LO16].

*Remark 1.1.* To the best of our knowledge, the first instance of Looijenga's framework is in Igusa's celebrated paper [Igu62] on modular forms of genus 2. The paper by Igusa analyzes the (birational) period map between the compactification of the moduli space of (smooth) genus 2 curves provided by the GIT quotient of binary sextics and the Satake compactification of  $\mathcal{A}_2$  (notice that  $\mathcal{A}_2$  is a locally symmetric variety of Type IV). Igusa describes explicitly the blow-up of a non-reduced point in the GIT moduli space needed to resolve the period map. See [Has05] for a more recent version of this story.

**1.1. The case of degree-2  $K3$  surfaces.** Let  $\mathcal{F}_2$  be the period space of degree-2 polarized  $K3$  surfaces, i.e.  $\mathcal{F}_2 = \Gamma_2 \backslash \mathcal{D}$ , where  $\mathcal{D}$  and  $\Gamma_2$  are defined as follows. Let  $\Lambda := U^2 \oplus E_8^2 \oplus A_1$ , where  $U$  is the hyperbolic plane, and root lattices are negative definite. Thus  $\Lambda$  is isomorphic to the primitive integral cohomology of a polarized  $K3$  of degree 2. Then

$$(1.1) \quad \mathcal{D} := \{[\sigma] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid q(\sigma) = 0, \quad q(\sigma + \bar{\sigma}) > 0\}^+, \quad \Gamma_2 := O^+(\Lambda).$$

Here the first superscript  $+$  means that we choose one connected component (there are two, interchanged by complex conjugation), the second one means that  $\Gamma_2$  is the index-2 subgroup of  $O(\Lambda)$  which maps  $\mathcal{D}$  to itself. Let  $\mathcal{F}_2 \subset \mathcal{F}_2^*$  be the Baily-Borel compactification. Let  $\mathfrak{M}_2 := |\mathcal{O}_{\mathbb{P}^2}(6)| // \text{PGL}(3)$  be the GIT moduli space of plane sextics. We let

$$\mathfrak{p}: \mathfrak{M}_2 \dashrightarrow \mathcal{F}_2^*, \quad \mathfrak{p}^{-1}: \mathcal{F}_2^* \dashrightarrow \mathfrak{M}_2$$

be the (birational) period map and its inverse, respectively. By Shah [Sha80] the period map  $\mathfrak{p}$  is regular away from the point  $q \in \mathfrak{M}_2$  parametrizing the  $\text{PGL}(3)$ -orbit of  $3C$ , where  $C \subset \mathbb{P}^2$  is a smooth conic (a closed orbit in  $|\mathcal{O}_{\mathbb{P}^2}(6)|^{ss}$ ). Let  $\mathfrak{M}_2^I \subset \mathfrak{M}_2$  be the open dense subset of orbits of curves with simple singularities, and let  $H_u \subset \mathcal{F}_2$  be the unigonal divisor, i.e. the divisor parametrizing periods of unigonal degree-2  $K3$ 's. Thus  $H_u$  is a Heegner divisor; it is the image in  $\mathcal{F}_2$  of a hyperplane  $v^\perp \cap \mathcal{D}$ , where  $v \in \Lambda$  has square  $-2$  and divisibility 2, i.e.  $(v, \Lambda) = 2\mathbb{Z}$  (any two such elements of  $v$  are  $\Gamma_2$ -equivalent). Then the period map defines an isomorphism  $\mathfrak{M}_2^I \xrightarrow{\sim} (\mathcal{F}_2 \setminus H_u)$ . Let  $L \in \text{Pic}(\mathfrak{M}_2)_{\mathbb{Q}}$  be the class induced by the hyperplane class on  $|\mathcal{O}_{\mathbb{P}^2}(6)|$ , let  $\lambda$  be the Hodge divisor class on  $\mathcal{F}_2$ , and  $\Delta := H_u/2$ ; a computation similar (but simpler) to those carried out in Sect. 4 of [LO16] gives that

$$(1.2) \quad \mathfrak{p}^{-1}L|_{\mathcal{F}_2} = \lambda + \frac{1}{2}H_u = \lambda + \Delta$$

(the  $\frac{1}{2}$  factor indicates that  $H_u$  is a ramification divisor). Arguing as in Sect. 4.2 of [LO16], one shows that  $\mathfrak{p}^{-1}$  is regular on all of  $\mathcal{F}_2$  (one key point is that  $\mathcal{F}_2$  is  $\mathbb{Q}$ -factorial). On the other hand  $\mathfrak{p}^{-1}$  is *not* regular on all of  $\mathcal{F}_2^*$ . In order to describe  $\mathfrak{p}^{-1}$  on the boundary of  $\mathcal{F}_2$ , let  $\widehat{\mathcal{F}}_2 \subset \mathcal{F}_2^* \times \mathfrak{M}_2$  be the closure of the graph of the restriction of  $\mathfrak{p}^{-1}$  to  $\mathcal{F}_2 \setminus H_u$ , and let  $\Pi: \widehat{\mathcal{F}}_2 \rightarrow \mathcal{F}_2^*$ ,  $\Phi: \widehat{\mathcal{F}}_2 \rightarrow \mathfrak{M}_2$

be the projections:

$$(1.3) \quad \begin{array}{ccc} & \widehat{\mathcal{F}}_2 & \\ \Pi \swarrow & & \searrow \Phi \\ \mathcal{F}_2^* & \xrightarrow{\mathfrak{p}^{-1}} & \mathfrak{M}_2 \end{array}$$

Thus  $\Pi$  is an isomorphism over  $\mathcal{F}_2$  (because  $\mathfrak{p}^{-1}$  is regular on  $\mathcal{F}_2$ ). On the other hand, it follows from Shah's description of semistable orbits in  $|\mathcal{O}_{\mathbb{P}^2}(6)|$ , that the fibers of  $\Pi$  over two of the four 1-dimensional boundary components of  $\mathcal{F}_2^*$  are 1-dimensional (namely those labeled by  $E_8^2 \oplus A_1$  and  $D_{16} \oplus A_1$ ; see Remark 5.6 of [Fri84] for the notation), and they are 0-dimensional over the remaining two boundary components. From this it follows that  $\mathcal{F}_2^*$  is not  $\mathbb{Q}$ -factorial, because if it were  $\mathbb{Q}$ -factorial, the exceptional divisor of  $\Pi$  would have pure codimension 1. Moreover, it follows that  $\Pi$  is a  $\mathbb{Q}$ -factorialization of  $\mathcal{F}_2^*$ . In fact, since  $\mathcal{F}_2$  is  $\mathbb{Q}$ -factorial, with  $\mathbb{Q}$  Picard group freely generated by  $\lambda$  and  $H_u$ , the  $\mathbb{Q}$  class group  $\text{Cl}(\mathcal{F}_2^*)_{\mathbb{Q}}$  is freely generated by  $\lambda^*$  and  $H_u^*$  (obvious notation). Since  $\lambda^*$  is the class of a  $\mathbb{Q}$ -Cartier divisor, it follows that  $H_u^*$  is not  $\mathbb{Q}$ -Cartier. Let  $\widehat{H}_u \subset \widehat{\mathcal{F}}_2$  be the strict transform of  $H_u$ . Then  $\widehat{H}_u$  is  $\mathbb{Q}$ -Cartier, because by (1.2), there exists  $m \gg 0$  such that  $m\widehat{H}_u$  is the divisor of a section of the line-bundle  $\Phi^*L^{2m} \otimes \Pi^*(\lambda^*)^{-2m}$ . Moreover we can identify  $\widehat{\mathcal{F}}_2$  with  $\text{Proj}(\bigoplus_{n \geq 0} \mathcal{O}_{\mathcal{F}_2^*}(nH_u^*))$ , because  $\widehat{H}_u$  is  $\Pi$ -ample (clearly  $a\pi^*(\lambda^*) + b\Phi^*L$  is ample for any  $a, b \in \mathbb{Q}_+$ , using (1.2) and the triviality of  $\pi^*(\lambda^*)$  on fibers of  $\Pi$ , it follows that  $\widehat{H}_u$  is  $\Pi$ -ample). Thus (as in [Loo86]) we have decomposed  $\mathfrak{p}^{-1}$  as follows: first we construct the  $\mathbb{Q}$ -factorialization of  $\mathcal{F}_2^*$  given by  $\text{Proj}(\bigoplus_{n \geq 0} \mathcal{O}_{\mathcal{F}_2^*}(nH_u^*))$ , then we blow down the strict transform of  $H_u^*$ , i.e.  $\widehat{H}_u$ . In this case the Mori chamber decomposition of the quadrant  $\{\lambda + \beta\Delta \mid \beta \in [0, 1] \cap \mathbb{Q}\}$  is very simple; there are exactly two walls, corresponding to  $\beta = 0$  and  $\beta = 1$ .

**1.2. A quick overview of Looijenga's framework.** Let  $\mathfrak{M}^0$  be a moduli space of (polarized) varieties which are smooth or “almost” smooth (e.g. surfaces with ADE singularities), with Hodge structure of  $K3$  type. In particular the corresponding period space is  $\mathcal{F} = \Gamma \backslash \mathcal{D}$ , where  $\mathcal{D}$  is a Type IV domain or a complex ball, and  $\Gamma$  is an arithmetic group. An example of  $\mathfrak{M}^0$  is provided by the moduli space of degree- $d$  polarized  $K3$  surfaces, embedded by a suitable multiple of the polarization (one also has to specify the linearized ample line-bundle on the relevant Hilbert scheme), and  $\mathcal{F} = \mathcal{F}_d$  - in particular the example discussed in **Subsection 1.1**. We let  $\mathfrak{M}^0 \subset \mathfrak{M}$  be a GIT compactification, and we let  $\mathcal{F} \subset \mathcal{F}^*$  be the Baily-Borel compactification. Let

$$\mathfrak{p}: \mathfrak{M} \dashrightarrow \mathcal{F}^*$$

be the period map, and assume that it is *birational*. Looijenga [Loo03a, Loo03b] tackled the problem of resolving  $\mathfrak{p}$ . First, he observed that in many instances  $\mathfrak{p}(\mathfrak{M}^0) = \mathcal{F} \setminus \text{supp } \Delta$ , where  $\Delta$  is an effective linear combination of Heegner divisors - in the example of **Subsection 1.1**, one chooses  $\Delta = H_u/2$ . It is reasonable to expect that

$$(1.4) \quad \mathfrak{M} \cong \text{Proj}R(\mathcal{F}, \lambda + \Delta),$$

where  $\lambda$  is the Hodge (automorphic) orbi-line bundle on  $\mathcal{F}$  (of course here the choice of coefficients for  $\Delta$  is crucial) - in the example of **Subsection 1.1**, Equation (1.4) holds by (1.2). On the other hand, Baily-Borel's compactification is characterized as

$$\mathcal{F}^* = \text{Proj}R(\mathcal{F}, \lambda).$$

Thus, in order to analyze the period map, we must examine  $\text{Proj}R(\mathcal{F}, \lambda + \beta\Delta)$  for  $\beta \in (0, 1) \cap \mathbb{Q}$  (we assume throughout that  $R(\mathcal{F}, \lambda + \beta\Delta)$  is finitely generated). Let us first consider the two



extreme cases:  $\beta$  close to 0 or to 1, that we denote  $\beta = \epsilon$  and  $\beta = (1 - \epsilon)$ , respectively. The space

$$\widehat{\mathcal{F}} := \text{Proj}R(\mathcal{F}, \lambda + \epsilon\Delta)$$

constructed by Looijenga [Loo03b] as a semi-toric compactification, has the effect of making  $\Delta$   $\mathbb{Q}$ -Cartier (the period space  $\mathcal{F}$  is  $\mathbb{Q}$ -factorial, the problems occur only at the Baily-Borel boundary). The map  $\widehat{\mathcal{F}} \rightarrow \mathcal{F}^*$  is a small map - in the example of **Subsection 1.1** this is the map  $\Pi: \widehat{\mathcal{F}}_2 \rightarrow \mathcal{F}_2^*$ . At the other extreme, we expect that  $\widetilde{\mathfrak{M}} := \text{Proj}R(\mathcal{F}, \lambda + (1 - \epsilon)\Delta)$  is a Kirwan type blow-up of the GIT quotient  $\mathfrak{M}$  which contracts the strict transform of  $\Delta$  - in the example of **Subsection 1.1** this is the map  $\Phi: \widehat{\mathcal{F}}_2 \rightarrow \mathfrak{M}_2$ .

In between, we expect a series of flips, dictated by the structure of the preimage of  $\Delta$  for the quotient map  $\pi: \mathcal{D} \rightarrow \mathcal{F}$ . More precisely, let  $\mathcal{H} := \pi^{-1}(\text{supp } \Delta)$ ; then  $\mathcal{H}$  is a union of hyperplane sections of  $\mathcal{D}$ , and hence is stratified by closed subsets, where a stratum is determined by the number of independent sheets (where “independent sheets” means that their defining equations have linearly independent differentials) of  $\mathcal{H}$  containing the general point of the stratum. The stratification of  $\mathcal{H}$  induces a stratification of  $\text{supp } \Delta$ , where the strata of  $\text{supp } \Delta$  are indexed by the “number of sheets” (in  $\mathcal{D}$ , *not* in  $\mathcal{F} = \Gamma \backslash \mathcal{D}$ ). Roughly speaking, Looijenga predicts that stratum of  $\text{supp } \Delta$  corresponding to  $k$  (at least) sheets meeting (in  $\mathcal{D}$ ) is flipped to a dimension  $k - 1$  locus on the GIT side. In the example of **Subsection 1.1**, the divisor  $\mathcal{H} := \pi^{-1}H_u$  is smooth, and this is the reason why no flips appear in the resolution of  $\mathfrak{p}$  given by (1.3). In **Subsection 1.3** we give an example in which one flip occurs.

In summary, Looijenga says that to resolve the inverse of the period map one has to follow the steps below:

- (1)  $\mathbb{Q}$ -factorialize  $\Delta$ .
- (2) Flip the strata of  $\Delta$  defined above, starting from the lower dimensional strata,
- (3) Contract the strict transform of  $\Delta$ .

All these operations have arithmetic origin, and thus, when applicable, give a meaningful stratification of the GIT moduli space.

**1.3. Cubic fourfolds.** The period space is defined similarly to that of degree-2 polarized  $K3$  surfaces (see (1.1)). Specifically,  $\Lambda$  is replaced by  $\Lambda' := U^2 \oplus E_8^2 \oplus A_2$ , and the arithmetic group is  $\widetilde{O}^+(\Lambda') := \widetilde{O}(\Lambda') \cap O^+(\Lambda')$ , where  $\widetilde{O}(\Lambda')$  is the stable orthogonal group. The divisor  $\Delta$  is  $H_u/2$ , where this time  $H_u$  is the image in  $\mathcal{F}$  of  $v^\perp \cap \mathcal{D}$  for  $v \in \Lambda$  such that  $q(v) = -6$  and the divisibility of  $v$  is 3, i.e.  $(v, \Lambda') = 3\mathbb{Z}$ . In this case at most two sheets of  $\mathcal{H} := \pi^{-1}H_u$  meet, and correspondingly there is exactly one flip  $f$ , fitting into the diagram

$$\begin{array}{ccc} \widehat{\mathcal{F}} & \xrightarrow{\quad f \quad} & \widetilde{\mathfrak{M}} \\ \Pi \downarrow & & \downarrow \Phi \\ \mathcal{F}^* & \xrightarrow{\quad \mathfrak{p}^{-1} \quad} & \mathfrak{M} \end{array}$$

Here,  $\Phi$  is the blow-up of the polystable point corresponding to the secant to Veronese. The map  $f$  is the flip of the codimension 2 locus where two sheets of  $\mathcal{H} := \pi^{-1}H_u$  meet, and the corresponding locus in  $\mathfrak{M}$  is the curve parametrizing cubic fourfolds singular along a rational normal curve. For a detailed treatment, see [Loo09, Laz09b, Laz10].

**1.4. Periods of polarized  $K3$ 's of degree 4 according to [LO16].** We start by recalling notation and constructions from [LO16]. For  $N \geq 3$ , let  $\Lambda_N := U^2 \oplus D_{N-2}$ . In [LO16] we defined a group  $\widetilde{O}^+(\Lambda_N) < \Gamma_N < O^+(\Lambda_N)$  which is equal to  $O^+(\Lambda_N)$  if  $N \not\equiv 6 \pmod{8}$ , and is of index 3 in

$O^+(\Lambda_N)$  if  $N \equiv 6 \pmod{8}$ , see Proposition 1.2.3 of [LO16]. Next, we let

$$(1.5) \quad \mathcal{D}_N := \{[\sigma] \in \mathbb{P}(\Lambda_{19} \otimes \mathbb{C}) \mid q(\sigma) = 0, \quad q(\sigma + \bar{\sigma}) > 0\}^+,$$

$$(1.6) \quad \mathcal{F}(N) := \Gamma_N \backslash \mathcal{D}_N.$$

(The meaning of the superscripts  $+$  is as in (1.1).) Then  $\mathcal{F} = \mathcal{F}(19)$  and  $\mathcal{F}_h = \mathcal{F}(18)$  are the period spaces for polarized  $K3$ 's of degree 4, and for hyperelliptic polarized  $K3$ 's of degree 4, respectively. Suppose that  $N \not\equiv 6 \pmod{8}$ ; the *hyperelliptic* divisor  $H_h(N) \subset \mathcal{F}(N)$  is the image of  $v^\perp \cap \mathcal{D}$  for  $v \in \Lambda_N$  such that  $q(v) = -4$ , and  $(v, \Lambda_N) = 2\mathbb{Z}$  (any two such  $v$ 's are  $O^+(\Lambda_N)$ -equivalent). If  $N \equiv 6 \pmod{8}$ , then the definition of the hyperelliptic divisor is subtler (there is a link with the fact that  $[O^+(\Lambda_N) : \Gamma_N] = 3$ ). The key aspect of our analysis in [LO16] is that we have a tower of locally symmetric spaces

$$(1.7) \quad \dots \hookrightarrow \mathcal{F}(18) \xrightarrow{f_{19}} \mathcal{F}(19) \xrightarrow{f_{20}} \mathcal{F}(20) \hookrightarrow \dots \hookrightarrow \mathcal{F}(N-1) \xrightarrow{f_N} \mathcal{F}(N) \hookrightarrow \dots$$

where  $\mathcal{F}(N-1)$  is embedded into  $\mathcal{F}(N)$  as the hyperelliptic divisor  $H_h(N)$ . Recall that  $H_h = H_h(19)$ . Let  $(X, L)$  be a polarized  $K3$  surface of degree 4, and  $\mathfrak{p}(X, L)$  its period point; then  $\mathfrak{p}(X, L) \in H_h$  if and only if  $(X, L)$  is hyperelliptic, i.e.  $\varphi_L : X \dashrightarrow |L|^\vee$  is a regular map of degree 2 onto a quadric - this explains our terminology.

The *unigonal* divisors in  $\mathcal{F}$ , denoted by  $H_u$ , is the image of  $v^\perp \cap \mathcal{D}$  for  $v \in \Lambda$  such that  $q(v) = -4$ , and  $(v, \Lambda_{19}) = 4\mathbb{Z}$  respectively (any two such  $v$ 's are  $O^+(\Lambda_{19})$ -equivalent). Let  $(X, L)$  be a polarized  $K3$  surface of degree 4, and let  $\mathfrak{p}(X, L)$  its period point; then  $\mathfrak{p}(X, L) \in H_u$  if and only if  $(X, L)$  is unigonal, i.e.  $L \cong \mathcal{O}_X(A + 3B)$ , where  $B$  is an elliptic curve and  $A$  is a section of the elliptic fibration  $|B|$ .

Of course, we do not “see” hyperelliptic polarized  $K3$ 's of degree 4 among quartic surfaces, nor do we see unigonal polarized  $K3$ 's of degree 4 - and that is where all the action takes place.

We let  $\Delta := (H_h + H_u)/2$ . For  $k \geq 1$ , let  $\Delta^{(k)} \subset \text{supp } \Delta$  be the  $k$ -th stratum of the stratification defined in **Subsection 1.2**, i.e. the closure of the image of the locus in  $\mathcal{H} := \pi^{-1}(\text{supp } \Delta)$  where  $k$  (at least) independent sheets of  $\mathcal{H}$  meet. One has  $\Delta^{(19)} \neq \emptyset$ , and there is a strictly increasing ladder  $\Delta^{(19)} \subsetneq \Delta^{(18)} \subsetneq \dots \subsetneq \Delta^{(1)} = (H_h \sqcup H_u)$ . This is in stark contrast with the cases discussed above: in fact (with analogous notation) in the case of degree 2  $K3$  surfaces one has  $\Delta^{(k)} = \emptyset$  for  $k \geq 2$ , and in the case of cubic fourfolds one has  $\Delta^{(k)} = \emptyset$  for  $k \geq 3$ . In fact, since for quartic surfaces there are 0-dimensional strata of  $\Delta$ , strictly speaking Looijenga's theory does not apply (see Lemma 8.1 in [Loo03b]). Our refinement [LO16] takes care of this issue and, at least to first order, Looijenga's framework still applies. We let  $\lambda = \lambda(19)$  be the Hodge  $\mathbb{Q}$  line-bundle on  $\mathcal{F}$ . The period map  $\mathfrak{p} : \mathfrak{M} \dashrightarrow \mathcal{F}^*$  (denoted  $\mathfrak{p}_{19}$  in [LO16]) is birational by Global Torelli, and it defines an isomorphism

$$\mathfrak{M} \cong \text{Proj} R(\mathcal{F}, \lambda + \Delta)$$

by Proposition 4.1.2 of [LO16]. On the other hand, the Baily-Borel compactification  $\mathcal{F}^*$  is identified with  $\text{Proj} R(\mathcal{F}, \lambda)$ . For  $\beta \in [0, 1] \cap \mathbb{Q}$ , we let

$$\mathcal{F}(\beta) = \text{Proj} R(\mathcal{F}, \lambda + \beta\Delta).$$

$$(1.8) \quad \begin{array}{ccccccc} \widehat{\mathcal{F}} \cong \mathcal{F}(0, \frac{1}{9}) & \dashrightarrow & \mathcal{F}(\frac{1}{9}, \frac{1}{7}) & \dashrightarrow & \dots & \mathcal{F}(\frac{1}{m+1}, \frac{1}{m}) & \dashrightarrow & \mathcal{F}(\frac{1}{m}, \frac{1}{m-1}) & \dots & \mathcal{F}(\frac{1}{3}, \frac{1}{2}) & \dashrightarrow & \mathcal{F}(\frac{1}{2}, 1) \cong \widetilde{\mathfrak{M}} \\ \downarrow \Pi & \searrow & \swarrow & & & \searrow & \swarrow & & & \searrow & \swarrow & \downarrow \Phi \\ & \mathcal{F}(\frac{1}{9}) & & \mathcal{F}(\frac{1}{7}) & & \mathcal{F}(\frac{1}{m}) & & \mathcal{F}(\frac{1}{2}) & & & & \mathcal{F}(1) \cong \mathfrak{M} \\ \mathcal{F}^* \cong \mathcal{F}(0) & & & & & & & & & & & \end{array}$$

The predictions of [LO16] are as follows. First, we expect that  $R(\mathcal{F}, \lambda + \beta\Delta)$  is finitely generated for all  $\beta \in [0, 1] \cap \mathbb{Q}$ , and that the critical values of  $\beta \in [0, 1] \cap \mathbb{Q}$  are given by

$$(1.9) \quad \beta \in \left\{0, \frac{1}{9}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}.$$

(Note:  $\beta = 1/8$  is missing, *no typo*.) This means that for  $\beta_i < \beta \leq \beta' < \beta_{i+1}$ , where  $\beta_i, \beta_{i+1}$  are consecutive critical values, the birational map  $\mathcal{F}(\beta) \dashrightarrow \mathcal{F}(\beta')$  is an isomorphism. We let

$$(1.10) \quad \mathcal{F}(\beta_i, \beta_{i+1}) := \mathcal{F}(\beta), \quad \beta \in [\beta_i, \beta_{i+1}] \cap \mathbb{Q}.$$

As we have already mentioned,  $\mathcal{F}(\epsilon)$  is expected to be the  $\mathbb{Q}$ -factorialization of  $\mathcal{F}^*$ . On the other hand,  $\mathcal{F}(1 - \epsilon)$  is the blow-up of  $\mathfrak{M}$  with center a scheme supported on the two points representing the tangent developable of a twisted cubic curve, and a double (smooth) quadric. For later reference we denote by  $v$  and  $\omega$  the corresponding points of  $\mathfrak{M}$ ; explicitly

$$(1.11) \quad v := [V(4(x_1x_3 - x_2^2)(x_0x_2 - x_1^2) - (x_1x_2 - x_0x_3)^2)],$$

$$(1.12) \quad \omega := [V((x_0^2 + x_1^2 + x_2^2 + x_3^2)^2)]$$

We note that the scheme structure at the point representing the tangent developable is *not* reduced. We predict that one goes from  $\mathcal{F}(\epsilon)$  to  $\mathcal{F}(1 - \epsilon)$  via a stratified flip, summarized in (1.8). More precisely, in [LO16] we have defined a tower of closed subsets

$$(1.13) \quad Z^9 \subset Z^8 \subset Z^7 \subset Z^5 \subset Z^4 \subset Z^3 \subset Z^2 \subset Z^1 = H_u \cup H_h \subset \mathcal{F},$$

where  $k$  denotes the codimension ( $Z^6$  is missing, *no typo*). In fact, with the notation of [LO16],

- for  $k \leq 5$ ,  $Z^k = \Delta^{(k)} = \text{Im}(f_{19-k,19}: \mathcal{F}(19-k) \hookrightarrow \mathcal{F})$ ,
- $Z^7 = \text{Im}(f_{13,19} \circ q_{13}: \mathcal{F}(\Pi_{2,10} \oplus A_2) \hookrightarrow \mathcal{F})$ ,
- $Z^8 = \text{Im}(f_{12,19} \circ m_{12}: \mathcal{F}(\Pi_{2,10} \oplus A_1) \hookrightarrow \mathcal{F})$ , and
- $Z^9 = \text{Im}(f_{11,19} \circ l_{11}: \mathcal{F}(\Pi_{2,10}) \hookrightarrow \mathcal{F})$  ( $Z^9$  is one of the two components of  $\Delta^{(9)}$ ).

Let  $m \in \{2, 3, \dots, 7, 9\}$ ; we predict that the birational map  $\mathcal{F}(a(m), \frac{1}{m}) \dashrightarrow \mathcal{F}(\frac{1}{m}, \frac{1}{m-1})$  (here  $a(m) = \frac{1}{m+1}$  if  $m \neq 7, 9$ ,  $a(7) = 1/9$ , and  $a(9) = 0$ ) is a flip with center the strict transform of (the closure) of  $Z^k$ , where  $k = m$ , except for  $m = 7, 6$ , in which case  $k = m + 1$ . Thus we expect that  $Z^k$  is replaced by a closed  $W_{k-1} \subset \mathfrak{M}$  of dimension  $k - 1$ . Correspondingly, we should have a stratification of the indeterminacy locus  $\text{Ind}(\mathfrak{p})$  of the period map. Now, according to Shah, the indeterminacy locus  $\text{Ind}(\mathfrak{p})$  is contained in the locus  $\mathfrak{M}^{IV}$  parametrizing polystable quadrics of Type IV (i.e. those which do not have slc singularities, see **Subsection 2.4**) - and it is natural to expect that  $\text{Ind}(\mathfrak{p}) = \mathfrak{M}^{IV}$ . The first evidence in favor of our predictions is that, as we will show,  $\mathfrak{M}^{IV}$  has a natural stratification

$$(1.14) \quad \mathfrak{M}^{IV} = (W_8 \sqcup \{v\}) \supset (W_7 \sqcup \{v\}) \supset (W_6 \sqcup \{v\}) \supset (W_4 \sqcup \{v\}) \supset (W_3 \sqcup \{v\}) \supset (W_2 \sqcup \{v\}) \supset (W_1 \sqcup \{v\}) \supset (W_0 \sqcup \{v\}),$$

where  $W_0 = \{\omega\}$ , and each  $W_i$  is (closed) irreducible of dimension  $i$ . It is well known that the period map improves on the blow-up  $\widetilde{\mathfrak{M}}$  of a certain subscheme of  $\mathfrak{M}$  supported on  $\{v, \omega\}$ . More precisely, it is regular on the exceptional divisor over  $v$ , with image the closure of the unigonal divisor  $H_u$ , it is regular on the dense open subset of the exceptional divisor over  $\omega$  parametrizing double covers of  $\mathbb{P}^1 \times \mathbb{P}^1$  ramified over a curve with ADE singularities, and it maps to  $H_u \setminus \Delta^{(2)}$ . This is discussed with much more detail than previously available in the literature (e.g. [Sha81]) in **Subsection 5.1** and **Subsection 5.2** respectively. In **Subsection 5.3** we identify  $\widetilde{\mathfrak{M}}$  with  $\mathcal{F}(1 - \epsilon)$ , for small  $\epsilon > 0$ . **Subsection 5.4** is devoted to a proof (long, without full details) that the blow up of a suitable scheme supported on the strict transform of  $W_1$  in  $\widetilde{\mathfrak{M}}$  can be contracted to produce  $\mathcal{F}(1/2)$ . Lastly, in **Subsection 5.5**, we give evidence that  $W_{k-1}$  is related to  $Z^k$  as predicted, for  $k \in \{7, 8, 9\}$ . Namely,  $Z^9, Z^8, Z^7$  correspond precisely to  $T_{2,3,7}, T_{2,4,5}, T_{3,3,4}$  marked  $K3$  surfaces respectively, while  $W_6, W_7, W_8$  correspond to the equisingular loci of quartics with

$E_{14}, E_{13}, E_{12}$  singularities respectively (on the GIT side). The flips replacing  $W_{k-1}$  with  $Z^k$  (in this range) are analogous to the semi-stable replacement that occurs for curves in the Hassett–Keel program (e.g. curves with cusps are replaced by stable curves with elliptic tails).

**1.5. Periods of hyperelliptic polarized  $K3$ 's of degree 4 according to [LO16].** As much as the GIT moduli space of quartic surfaces does not see hyperelliptic quartic  $K3$ 's, the moduli space  $\mathfrak{M}_h$  does not see “hyperelliptic quartic  $K3$ 's”, i.e.  $(X, L)$  such that  $\varphi_L(X)$  is a quadric cone. The periods of this kind of hyperelliptic quartics are parametrized by the “hyperelliptic” divisor  $H_h(18)$ . The boundary divisor for  $\mathcal{F}_h$  is given by  $\Delta_h := H_h(18)/2$  (the analogue in  $\mathcal{F}_h$  of the unigonal divisor  $H_u$  in  $\mathcal{F}$  is 0, see [LO16]).

Let  $\mathfrak{p}_h: \mathfrak{M}_h \dashrightarrow \mathcal{F}_h$  be the period map obtained by associating to a smooth  $(4, 4)$ -curve  $D$  the period point of the (polarized) double cover of  $\mathbb{P}^1 \times \mathbb{P}^1$  ramified over  $D$ . We predict that the inverse period map  $\mathfrak{p}_h^{-1}$  will factor as a product of elementary birational modifications, analogous to those discussed above for  $\mathfrak{p}^{-1}$ , and that they correspond the jumps in the models  $\mathcal{F}_h(\beta) := \text{Proj} R(\mathcal{F}_h, \lambda + \beta \Delta_h)$  for  $\beta \in [0, 1] \cap \mathbb{Q}$ . In particular, the results of [LO16] say that the intermediate critical slopes should be

$$(1.15) \quad \beta \in \left\{ \frac{1}{8}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \right\}$$

(same as for quartics, except for a shift by 1 in the denominator). Furthermore, the predicted centers of flips are the restrictions of the centers for quartics to the hyperelliptic locus. The analogue of (1.13) is the tower

$$(1.16) \quad Z_h^8 \subset Z_h^7 \subset Z_h^6 \subset Z_h^4 \subset Z_h^3 \subset Z_h^2 \subset Z_h^1 \subset \mathcal{F}_h,$$

where  $Z_h^k = Z^{k+1} \cap \mathcal{F}_h$  (recall that  $\mathcal{F}(18) = H_h(19)$ , see (1.7)).

On the other side (see **Subsection 3.4**), there is a natural analogue of (1.14):

$$(1.17) \quad \mathfrak{M}_h^{IV} = W_{h,7} \supset W_{h,6} \supset W_{h,5} \supset W_{h,3} \supset W_{h,2} \supset W_{h,1} \supset W_{h,0},$$

where each  $W_{i,h}$  is (closed) irreducible of dimension  $i$ . (In the hyperelliptic case, the analogue of  $v$  is missing.) Our prediction, as for quartics, is that at a critical value  $\beta \in \left\{ \frac{1}{8}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \right\}$  the strict transform of (the closure of)  $Z_h^k$  is replaced by  $W_{h,k-1}$ .

## 2. GIT AND HODGE-THEORETIC STRATIFICATIONS OF $\mathfrak{M}$ AND $\mathfrak{M}_h$

**2.1. Summary.** The analysis of GIT (semi)stability for quartic surfaces was carried out by Shah in [Sha81]. In this section we will review some of his results. In particular we will go over the GIT stratification (N.B. as usual, a *stratification* of a topological space  $X$  is a partition of  $X$  into locally closed subsets such that the closure of a stratum is a union of strata) determined by the stabilizer groups of polystable quartics. After that, we will review Shah's Hodge-theoretic stratification [Sha81, Sha79]

$$(2.1) \quad \mathfrak{M} = \mathfrak{M}^I \sqcup \mathfrak{M}^{II} \sqcup \mathfrak{M}^{III} \sqcup \mathfrak{M}^{IV}$$

from a modern perspective (due to Steenbrink [Ste81], Kollár, Shepherd-Barron and others [KSB88], [SB83a], [KK10]). The period map  $\mathfrak{p}: \mathfrak{M} \dashrightarrow \mathcal{F}^*$  extends regularly away from  $\mathfrak{M}^{IV}$ , and it maps  $\mathfrak{M}^I$ ,  $\mathfrak{M}^{II}$  and  $\mathfrak{M}^{III}$  to the interior  $\mathcal{F}$ , to the union of the Type II boundary components, and to the Type III locus (a single point) respectively. A posteriori, we will see that  $\mathfrak{M}^{IV}$  is equal to the indeterminacy of the period map.

Similarly, Shah carried out the analysis of GIT (semi)stability for  $(4, 4)$ -curves on  $\mathbb{P}^1 \times \mathbb{P}^1$  (for the action of  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ ) in [Sha81]. We will quickly go over Shah's (semi)stability analysis.

Then we will describe the stratification of  $\mathfrak{M}_h$  determined by the stabilizer groups of polystable  $(4, 4)$ -curves, and the Hodge-theoretic stratification

$$(2.2) \quad \mathfrak{M}_h = \mathfrak{M}_h^I \sqcup \mathfrak{M}_h^{II} \sqcup \mathfrak{M}_h^{III} \sqcup \mathfrak{M}_h^{IV}$$

analogous to (2.1). We should note that the fiber of the blow-up of  $\mathfrak{M}$  with center the (reduced) point  $\omega$  is isomorphic to  $\mathfrak{M}_h$ , and that by gluing the GIT and Hodge-theoretic stratifications of  $\mathfrak{M} \setminus \{\omega\}$  and  $\mathfrak{M}_h$  we will get a stratification of the blow-up.

Much of this paper is concerned with the behaviour of the period map for quartic surfaces along  $\mathfrak{M}^{IV}$ , and for double covers of  $\mathbb{P}^1 \times \mathbb{P}^1$  ramified over a  $(4, 4)$  curve along  $\mathfrak{M}_h^{IV}$ .

**Notation/Convention 2.1.** Shah also defined a refinement of the stratification in (2.1), see Theorem 2.4 of [Sha81]. We will follow the notation of Theorem 2.4 of [Sha81], with an S prefix, and with the symbol IV replacing “Surfaces with significant limit singularities”. Thus the strata will be denoted by S-I, S-II(A,i), S-II(A,ii), S-III(B,ii), S-IV(A,i), etc. We recall that the roman numerals I, II, III, IV refer to the stratum of (2.1) to which a stratum belongs, and the letter A (B) indicates whether the stratum is contained in the stable locus or in the properly semistable locus. We will refer to Shah’s stratification before discussing the stratification in (2.1); this is not an issue, because the strata are defined explicitly by Shah in terms of singularities, see Theorem 2.4 of [Sha81].

The same notations and conventions apply to the case of hyperelliptic quartics. The relevant stratification of Shah is given in Theorem 4.8 of [Sha81].

**2.2. The GIT (or Kirwan) stratification for quartic surfaces.** Shah [Sha81] essentially established a relation between GIT (semi)stability of a quartic surface and the nature of its singularities. In particular he proved that a quartic with ADE singularities is stable, and hence there is an open dense subset  $\mathfrak{M}^I \subset \mathfrak{M}$  parametrizing isomorphism classes of polarized  $K3$  surfaces  $(X, L)$  such that  $L$  is very ample, i.e.  $(X, L)$  is neither hyperelliptic, nor unigonal. We will elaborate further on this in **Subsection 2.4**. In the present subsection the focus is on stabilizers (in  $\mathrm{SL}(4)$ ) of polystable quartics, and the stratification of  $\mathfrak{M}$  that they define (a quartic is *polystable* if its  $\mathrm{PGL}(4)$ -orbit is closed in the semistable locus  $|\mathcal{O}_{\mathbb{P}^3}(4)|^{ss}$ ). The point of view is essentially due to Kirwan [Kir85, Kir89]. Let  $\mathfrak{M}^s \subset \mathfrak{M}$  be the open dense subset parametrizing isomorphism classes of GIT stable quartics. Points of the *GIT boundary*  $\mathfrak{M} \setminus \mathfrak{M}^s$  parametrize isomorphism classes of polystable strictly semistable quartics. The stabilizer of such an orbit is a positive dimensional reductive subgroup. The classification of 1-dimensional stabilizers leads to the decomposition of the GIT boundary into irreducible components.

**Lemma 2.2.** *Let  $X = V(f)$  be a strictly polystable quartic. Then, with an appropriate choice of coordinates,  $f$  is stabilized by one of the following four 1-PS’s of  $\mathrm{SL}(4)$ :*

$$\lambda_1 = (3, 1, -1, -3), \quad \lambda_2 = (1, 0, 0, -1), \quad \lambda_3 = (1, 1, -1, -1), \quad \lambda_4 = (3, -1, -1, -1).$$

For  $i = 1, \dots, 4$ , let  $\sigma_i \subset \mathfrak{M}$  be the closed subset parametrizing polystable points stabilized by  $\lambda_i$ . Then the following hold:

- (1)  $\sigma_1, \dots, \sigma_4$  are the irreducible components of the GIT boundary  $\mathfrak{M} \setminus \mathfrak{M}^s$ .
- (2) The  $\sigma_i$ ’s are related to Shah’s stratification as follows:
  - $\sigma_1$  is the closure of the  $\tilde{E}_8$  component of S-II(B,i),
  - $\sigma_2$  is the closure of the  $\tilde{E}_7$  component of S-II(B,i) (N.B.  $\overline{S-II(B,i)} = \sigma_1 \cup \sigma_2$ ),
  - $\sigma_3 = \overline{S-II(B,ii)}$ ,
  - $\sigma_4 = \overline{S-II(B,iii)}$ .
- (3)  $\dim \sigma_1 = 2$ ,  $\dim \sigma_2 = 4$ ,  $\dim \sigma_3 = 2$ , and  $\dim \sigma_4 = 1$ .



*Proof.* This follows from Proposition 2.2 of [Sha81] (see also Kirwan [Kir89, §4] for a discussion focused on stabilizers). Specifically, the first 3 cases correspond to 1-PS subgroups of type  $(n, m, -m, -n)$  (i.e. Case (1) in loc. cit.). Thus  $\lambda_1, \lambda_2, \lambda_3$  correspond to (1.1), (1.2), and (1.3) respectively in Shah's analysis. The last case,  $\lambda_4$  corresponds to the cases (3.1) or (4.1) of Shah (N.B. the two cases are dual, so they result in a single case in our lemma; the previous case (1) is self-dual). It is easy to see that the other cases in Shah's analysis can be excluded (i.e. either they lead to unstable points, or to cases that are already covered by one of  $\lambda_1, \dots, \lambda_4$  – it is possible to have a polystable orbit stabilized by another 1-PS  $\lambda$ , but then the stabilizer contains a higher dimensional torus, which in turn contains a conjugate of one of  $\lambda_1, \dots, \lambda_4$ ). In conclusion, the GIT boundary consists of the 4 boundary components  $\sigma_i$  as stated (they intersect, but none is included in another).

Item (B) of Theorem 2.4 of Shah [Sha81] describes the strictly polystable locus in the GIT compactification. It is clear (from the geometric description and proofs) that the strictly semistable locus in  $\mathfrak{M}$  is the closure of the Type II strata, i.e.

$$\mathfrak{M} \setminus \mathfrak{M}^s = \cup_{i=1,4} \sigma_i = \overline{\text{S-II(B,i)}} \cup \overline{\text{S-II(B,ii)}} \cup \overline{\text{S-II(B,iii)}}.$$

Finally, the stratum S-II(B,i) has two components corresponding to quartics with two  $\tilde{E}_8$  singularities and two  $\tilde{E}_7$  singularities respectively (see [Sha81, Thm. 2.4 (B, II(i))]) for precise definitions of the two cases).

In order to compute the dimensions, one can write down normal forms for the quartics stabilized by the 1-PS  $\lambda_i$ . For instance, it is immediate to see that a quartic stabilized by  $\lambda_4 = (3, -1, -1, -1)$  is of the form  $x_0 f_3(x_1, x_2, x_3)$  (i.e. the union of the cone over a cubic curve with a transversal hyperplane, or same as S-II(B,iii)). Furthermore, we can still act on this equation with the centralizer of  $\lambda_4$  in  $\text{SL}(4)$ . In particular, with  $\text{SL}(3)$  acting on the variables  $(x_1, x_2, x_3)$ . It follows that the dimension in  $\mathfrak{M}$  of the locus of polystable points with stabilizer  $\lambda_4$  (i.e.  $\sigma_4$ ) is 1. At the other extreme, we have the case  $\lambda_1 = (3, 1, -1, -3)$ . In this case, the centralizer is the maximal torus in  $\text{SL}(4)$ . There are five degree 4 monomials stabilized by  $\lambda_1$ , namely  $x_0 x_2^3, x_1^3 x_3, (x_0 x_3)^a (x_1 x_2)^b$  with  $a + b = 2$ . It follows that  $\dim \sigma_1 = 2$ . (As noted above, a generic quartic stabilized by  $\lambda_4$  will have two  $\tilde{E}_8$  singularities, at  $[1, 0, 0, 0]$  and  $[0, 0, 0, 1]$  respectively, i.e. it is a sub-case of S-II(B,i).) The other cases are similar.  $\square$

Note that  $\sigma_1, \dots, \sigma_4$  are closed subsets of  $\mathfrak{M}$ . As a general rule, subsets of  $\mathfrak{M}$  denoted by Greek letters are closed.

The intersections of the components of the GIT boundary are determined by considering stabilizers that are tori of dimension larger than 1. Finally, special strata inside the  $\sigma_i$  are determined by other reductive (non-tori) stabilizers. The stratification of GIT quotients in terms of stabilizer subgroups plays an essential role in the work of Kirwan [Kir85], and the case of hypersurfaces of low degree was analyzed in [Kir89]. The following is due to Kirwan (and essentially contained also in [Sha81]).

**Proposition 2.3** (Kirwan [Kir89, §6]). *Let  $X = V(f)$  be a properly semistable polystable quartic. Then the connected component of the identity in the stabilizer of  $f$  in  $\text{SL}(4)$  is one of the following:*

- (1) *The trivial group  $\{1\}$  (i.e.  $X$  is stable).*
- (2) *One of the 1-PS's  $\lambda_1, \dots, \lambda_4$  listed above.*
- (3) *The two-dimensional torus  $\text{diag}(s, t, t^{-1}, s^{-1}) \subset \text{SL}(4, \mathbb{C})$ . Equivalently,  $X = Q_1 + Q_2$  where  $Q_1, Q_2$  are smooth quadrics meeting along 2 pairs of skew lines (special case of S-III(B,ii)). Let  $\tau \subset \mathfrak{M}$  be the closure of the set of points representing such quartics. Then  $\tau$  is a curve, and*

$$\tau = \sigma_1 \cap \sigma_2 \cap \sigma_3$$

*(and in fact  $\tau$  is the intersection of any two of the  $\sigma_1, \sigma_2, \sigma_3$ ).*

- (4) The maximal torus in  $\mathrm{SL}(4, \mathbb{C})$ . Equivalently,  $X$  is a tetrahedron ( $S\text{-III}(B, i)$ ). We let  $\zeta \in \mathfrak{M}$  be the corresponding point. Then

$$\{\zeta\} = \sigma_1 \cap \sigma_2 \cap \sigma_3 \cap \sigma_4.$$

- (5)  $\mathrm{SO}(3, \mathbb{C})$ , or equivalently  $X = Q_1 + Q_2$  where  $Q_1, Q_2$  are quadrics tangent along a smooth conic ( $S\text{-IV}(B, ii)$ ). This defines a curve  $\chi \subset \sigma_2 \subset \mathfrak{M}$ . The only incidence with the other strata is  $\chi \cap \tau = \{\omega\}$ .
- (6)  $\mathrm{SL}(2, \mathbb{C})$  (embedded in  $\mathrm{SL}(4)$  via the  $\mathrm{Sym}^3$ -representation). Equivalently,  $X$  is the tangent developable of a twisted cubic curve (special case of  $S\text{-IV}(B, i)$ ). Let  $v \in \mathfrak{M}$  be the corresponding point. One has  $v \in \sigma_1$ , and  $v \notin \sigma_i$  for  $i \in \{2, 3, 4\}$ .
- (7)  $\mathrm{SO}(4, \mathbb{C})$ . Equivalently,  $X = 2Q$ , where  $Q$  is a smooth quadric ( $S\text{-IV}(B, iii)$ ). We denote by  $\omega \in \mathfrak{M}$  the corresponding point. It holds,  $\omega \in \tau$  (see Remark below), and thus  $\omega \in \sigma_1 \cap \sigma_2 \cap \sigma_3$  (and  $\omega \notin \sigma_4$ ).

*Remark 2.4.* We consider the case of two quadrics meeting in two pairs of skew lines (case  $\tau$  above). It is easy to see that the associated general equation is

$$(a_1 x_0 x_3 + b_1 x_1 x_2)(a_2 x_0 x_3 + b_2 x_1 x_2) = 0.$$

Clearly this is a pencil, and we have the following two special cases:

- the tetrahedron (case  $\zeta$ ) if any of the  $a_i$  or  $b_i$  vanish;
- the double quadric (case  $\omega$ ) if  $[a_1, b_1] = [a_2, b_2] \in \mathbb{P}^1$ .

If both  $a_i$  or  $b_i$  vanish simultaneously, the associated quartic is unstable and thus the two cases above are distinct.

*Remark 2.5.* The general equation for the case of two quadrics tangent along a conic (case  $\chi$  above) is

$$f_{a,b} := (q(x_0, x_1, x_2) + ax_3^2)(q(x_0, x_1, x_2) + bx_3^2) = 0,$$

for  $[a, b] \in \mathbb{P}^1$  (N.B. if  $a = b = 0$ , one gets the double quadric cone, which is unstable; similarly, if  $a = \infty$  or  $b = \infty$ , one gets an unstable quadric). Note that  $f_{a,a} = 0$  is the equation of the double (smooth) quadric (case  $\omega$ ).

**2.3. The GIT (or Kirwan) stratification for  $(4, 4)$ -curves on  $\mathbb{P}^1 \times \mathbb{P}^1$ .** We have defined  $\mathfrak{M}_h$  as the GIT quotient for  $(4, 4)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$  (N.B. from our perspective  $\mathfrak{M}_h$  is an extremal birational model for the moduli of hyperelliptic quartic  $K3$  surfaces). As already mentioned,  $\mathfrak{M}_h$  can be identified with the exceptional divisor of the Kirwan blow-up of the point  $\omega$  (corresponding to the double quadric) in  $\mathfrak{M}$  (this follows [Sha80, Section 4], and it is discussed in more detail in **Subsection 5.2** below, see especially **Proposition 5.5**). Thus, the structure of the GIT strata in  $\mathfrak{M}_h$  is essentially the same as that of the GIT strata incident to the point  $\omega$ . Thus, the relevant stabilizers from **Proposition 2.3** are those that can factor through  $\mathrm{SL}(2) \times \mathrm{SL}(2) \subset \mathrm{SL}(4)$  (coming from the Segre embedding  $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ ). Thus, the relevant cases will be the 1-PS  $\lambda_1, \lambda_2, \lambda_3$  (but not  $\lambda_4$ ), and the items (3) and (5) of **Proposition 2.3** (missing (4) and (6) because of non-incidence, and (7) is eliminated by blow-up).

Shah [Sha81, Proposition 4.5] discusses the three cases with 1-parameter stabilizers. From our perspective, these 3 strata represent the traces of the strata  $\sigma_i \in \mathfrak{M}$  in  $\mathfrak{M}_h$  for  $i = 1, 2, 3$  (i.e. blow-up  $\omega \in \mathfrak{M}$  as described in **Subsection 5.2**, then consider the restriction of the strict transform of  $\sigma_i$  to the exceptional divisor, which is then identified with  $\mathfrak{M}_h$ ).  $\mathfrak{M}_h$  parameterizes  $(4, 4)$  curves on  $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ . One obtains the following analogue of **Proposition 2.3**:

**Proposition 2.6.** *The strictly polystable  $(4, 4)$  curves  $C$  are of one of the following types:*

- i) (Stabilizer  $\lambda_1$ )  $C$  decomposes as sum of two lines and two twisted cubics (i.e.  $(4, 4) = (1, 0) + (1, 0) + (1, 2) + (1, 2)$ ) such that the twisted cubics are tangent to the two lines in the same points (i.e. all the intersections occur at 2 points, one on  $L_1$  and one on  $L_2$ ). In this situation, generically, we have 2 singularities of type  $\tilde{E}_8$  at the two intersection points.
- ii) (Stabilizer  $\lambda_2$ )  $C$  decomposes as four conics sharing an axis (i.e. all meet at the same 2 points). In this case, generically, there are 2 singularities of type  $\tilde{E}_7$ .
- iii) (Stabilizer  $\lambda_3$ )  $C$  decomposes as two skew lines with multiplicity 2 and 4 other lines.
- iv) (Stabilizer the maximal torus in  $\mathrm{SL}(2) \times \mathrm{SL}(2)$ )  $C$  decomposes as two pairs of skew lines, each with multiplicity 2.
- v) (Stabilizer  $\mathrm{SO}(3, C) \subset \mathrm{SO}(4, \mathbb{C})$ )  $C$  is a conic with multiplicity 4.

**2.4. The stratification by Type.** Shah [Sha79], influenced by Mumford, defined the concept of “*insignificant limit singularity*”, and used it to study the period map for degree 2 and degree 4  $K3$  surfaces (see [Sha80, Sha81]). One defines  $\mathfrak{M}^{IV}$  as the subset of  $\mathfrak{M}$  parametrizing quartics with *significant* limit singularities. The main point is that the restriction of the period map to  $(\mathfrak{M} \setminus \mathfrak{M}^{IV})$  is regular. Hence the stratification  $\mathcal{F}^* = \mathcal{F} \sqcup \mathcal{F}^{II} \sqcup \mathcal{F}^{III}$ , where  $\mathcal{F}^{II}$  is the union of the Type II boundary components, and  $\mathcal{F}^{III}$  is the (unique) Type III boundary component, defines (by pull-back via  $\mathfrak{p}$ ) the strata  $\mathfrak{M}^I$ ,  $\mathfrak{M}^{II}$  and  $\mathfrak{M}^{III}$  (of course  $\mathfrak{M}^I$  coincides with the set that we had already defined).

*Remark 2.7.* The same considerations on Type stratification apply to the hyperelliptic case. In particular, we have a stratification  $\mathfrak{M}_h = \mathfrak{M}_h^I \sqcup \mathfrak{M}_h^{II} \sqcup \mathfrak{M}_h^{III} \sqcup \mathfrak{M}_h^{IV}$ . We only specific thing to notice in the hyperelliptic case is that we frequently pass back and forth between the singularities of hyperelliptic surface  $X$  and those of the ramification curve  $C$ . More precisely, we have  $X \xrightarrow{2:1} Q \subset \mathbb{P}^3$  with ramification curve  $C$ . Assume for simplicity  $Q$  is smooth (this is the case at the level of  $\mathfrak{M}_h$ ; as we interpolate from  $\mathfrak{M}_h$  to  $\mathcal{F}_h^*$ ,  $Q$  will be allowed to be singular, but a similar discussion applies there). Then the singularities of  $X$  and those of  $C$  are in one-to-one correspondence including the type up to suspension (N.B. recall that two hypersurface singularities  $V(f(x_1, \dots, x_n))$  and  $V(g(x_1, \dots, x_{n+k}))$  have the same type, if in appropriate coordinates,  $g = f + x_{n+1}^2 + \dots + x_{n+k}^2$ ). Thus, the classification in types refers to the singularities of  $X$ , but then it might be convenient to refer to the singularities of  $C$  (e.g. having Type I is the same thing as saying that  $C$  has ADE singularities).

We will start by giving an updated view of the concept of insignificant limit singularity. Briefly, Steenbrink [Ste81] noticed that an insignificant limit singularity is du Bois. On a different track, from the perspective of moduli, Shepherd-Barron [SB83a] and then Kollár–Shepherd-Barron [KSB88] noticed that the right notion of singularities is that of semi-log-canonical (slc) singularities. More recently (with [KK10] as the last step), it was proved that an slc singularity is du Bois. Lastly, one can check by direct inspection that Shah’s list of insignificant singularities coincides with the list of Gorenstein slc surface singularities (which are then du Bois). Of course, in the situation studied here, this is just a long-winded highbrow reproof of Shah’s results from 1979, but what is gained is a conceptual understanding of the situation.

We should also point out the connection between slc singularities and GIT. On one hand, an easy observation ([Hac04], [KL04]) shows that a quartic with slc singularities is GIT semistable. A much deeper result (due to Odaka [Oda12, Oda13]), which can be viewed as some sort of converse of this, is giving a close connection between slc singularities and  $K$ -stability. Finally,  $K$ -stability should be viewed as a refined notion of asymptotic stability. We caution however that the precise connection between asymptotic stability and  $K$ -stability/slc for  $K3$  surfaces is not known. More precisely, an example of Shepherd-Barron [SB83a, SB83b] shows that for  $K3$ s of big enough degree there is no (usual) asymptotic GIT stability. The results of [WX14] strengthen the meaning of this

failure of asymptotic stability. Nonetheless, it is still possible that a certain (weaker) asymptotic stabilization exists. We hope that our HKL program will eventually address this issue.

**2.4.1. ADE singularities.** We recall that  $\mathfrak{M}^I \subset \mathfrak{M}$  is (by definition) the subset parametrizing isomorphism classes of quartics with ADE singularities. The following identification of  $\mathfrak{M}^I$  (as a quasi-projective variety) with an open subset of the projective variety  $\mathcal{F}^*$  is well known:

**Theorem 2.8.** *The period map defines an isomorphism*

$$\mathfrak{M}^I \xrightarrow{\sim} (\mathcal{F} \setminus H_h \setminus H_u).$$

**2.4.2. Insignificant Limit Singularities.** We recall the following important result about slc singularities.

**Theorem 2.9** (Kollár–Kovács [KK10], Shah [Sha79] (for dimension 2)). *Let  $X_0$  be a projective reduced variety (not necessarily irreducible) with slc singularities. Then  $X$  has du Bois singularities. In particular, if  $\mathcal{X}/B$  is a smoothing of  $X_0$  over a pointed smooth curve  $(B, 0)$ , then the natural map  $H^n(X_0) \rightarrow H_{\lim}^n$  induces an isomorphism*

$$I^{p,q}(X_0) \cong I_{\lim}^{p,q}$$

*on the  $I^{p,q}$  components of the MHS with  $p \cdot q = 0$ .*

The key point (for us) of the above result is that, if the generic fiber of  $\mathcal{X}/B$  is a (smooth) K3 surface, then the MHS of the central fiber  $X_0$  essentially determines the limit MHS associated to  $\mathcal{X}^*/(B \setminus \{0\})$ . This is a result due to Shah [Sha79] in dimension 2 and Gorenstein singularities (the case relevant for us). Steenbrink [Ste81] connected this result to the notion of du Bois singularities.

**Definition 2.10.** A reduced (not necessarily irreducible) projective surface  $X_0$  is a *degeneration of K3 surfaces* if it is the central fiber of a flat proper family  $\mathcal{X}/B$  over a pointed smooth curve  $(B, 0)$  such that  $\omega_{\mathcal{X}/B} \equiv 0$  and the general fiber  $X_b$  is a smooth K3 surface. We say that  $X_0$  has *insignificant limit singularities* if  $X_0$  has semi-log-canonical singularities.

*Remark 2.11.* The list of singularities baptized as *insignificant limit singularities* by Shah [Sha79] coincides with the list of Gorenstein slc singularities (see [SB83a], [KSB88]). For a degeneration of K3 surfaces, the Gorenstein assumption is automatic.

Let  $X_0$  be a degeneration of K3 surfaces with insignificant singularities. On  $H^2(X_0)$  we have a MHS of weight 2. Denote by  $h^{p,q}$  the associated Hodge numbers ( $h^{p,q} = \dim_{\mathbb{C}} I^{p,q}$ ). **Theorem 2.9** gives that one, and only one, of the following 3 equalities holds:

- $h^{2,0}(X_0) = 1$ .
- $h^{1,0}(X_0) = 1$ .
- $h^{0,0}(X_0) = 1$ .

In fact this follows from the isomorphism of the theorem, and the fact that  $h_{\lim}^{2,0} + h_{\lim}^{1,0} + h_{\lim}^{0,0} = 1$  for a degeneration of K3's.

**Definition 2.12.** Let  $X_0$  be a degeneration of K3s.

- (1)  $X_0$  has *Type I* if it has insignificant limit singularities, and  $h^{2,0}(X_0) = 1$ .
- (2)  $X_0$  has *Type II* if it has insignificant limit singularities, and  $h^{1,0}(X_0) = 1$ .
- (3)  $X_0$  has *Type III* if it has insignificant limit singularities, and  $h^{0,0}(X_0) = 1$ .
- (4)  $X_0$  has *Type IV* if it has significant limit singularities.

We are interested in the case of Gorenstein slc surfaces. These are classified by Kollár–Shepherd-Barron [KSB88] and Shepherd-Barron. They are

- (A) ADE singularities (canonical case)

- (B) simple elliptic singularities (for hypersurfaces the relevant cases are  $\tilde{E}_r$  with  $r = 6, 7, 8$ ), surfaces singular along a curve, generically normal crossings (or equivalently  $A_\infty$  singularities) and possibly ordinary pinch points (aka  $D_\infty$ ).
- (C) cusp and degenerate cusp singularities.

*Remark 2.13.* We note that a normal crossing degeneration without triple points is a Type II degeneration, while a normal crossing degeneration with triple points is a Type III degeneration (a triple point is a particular degenerate cusp singularity).

By applying results of Shah [Sha79] and Kulikov-Persson-Pinkham's Theorem (see also Shepherd-Barron [SB83a]), one obtains the following.

**Theorem 2.14.** *Let  $X_0$  be a degeneration of K3 surfaces with insignificant singularities. Then the following hold:*

- i)  $X_0$  is of Type I if and only if it has ADE singularities,
- ii) if  $X_0$  is of Type II then, with the exception of rational double (i.e. ADE) points,  $X_0$  has a simple elliptic singularity or it is singular along a curve which is either smooth elliptic (and has no pinch points), or rational with 4 pinch points. (N.B. all non-ADE singularities are of this type, and at least one occurs.)
- iii) if  $X_0$  is of Type III then, with the exception of ADE and  $A_\infty$  singularities, all singularities of  $X_0$  are either cusp or degenerate cusps, and at least one of these occurs.

#### 2.4.3. The stratification and the period map.

**Proposition 2.15.** *Let  $X_0$  be a quartic surface with insignificant singularities. Then  $X_0$  is GIT semistable.*

*Proof.* This follows from the general fact observed by Hacking and Kim-Lee [KL04] (see esp. the proof of Proposition 10.2 in [Hac04]): GIT (semi)stability (via the numerical criterion) and the log canonical threshold are computed via the same recipe, with the difference that in the case of GIT (semi)stability one allows only linear changes of coordinates (vs. analytic in the other case). Thus, the inequality needed for log canonicity implies the inequality needed for semistability. The result also follows by inspection from Shah [Sha81] (i.e. an unstable quartic does not have slc singularities).  $\square$

**Definition 2.16.** We let  $\mathfrak{M}^I, \mathfrak{M}^{II}, \mathfrak{M}^{III} \subset \mathfrak{M}$  be the subsets of points represented by polystable quartics with insignificant limit singularities of Type I, Type II and Type III respectively (note that  $\mathfrak{M}^I$  is the same subset as the previously defined  $\mathfrak{M}^I$ , by **Theorem 2.14**). We let  $\mathfrak{M}^{IV} \subset \mathfrak{M}$  be the subset of points represented by polystable quartics with significant limit singularities.

Below is the result that was described at the beginning of the present section.

**Proposition 2.17.**  $\mathfrak{M}^I, \mathfrak{M}^{II}, \mathfrak{M}^{III}, \mathfrak{M}^{IV}$  define a stratification of  $\mathfrak{M}$ . The period map  $\mathfrak{p}: \mathfrak{M} \rightarrow \mathcal{F}^*$  is regular away from  $\mathfrak{M}^{IV}$ , and

$$\mathfrak{p}(\mathfrak{M}^I) \subset \mathcal{F}, \quad \mathfrak{p}(\mathfrak{M}^{II}) \subset \mathcal{F}^{II}, \quad \mathfrak{p}(\mathfrak{M}^{III}) \subset \mathcal{F}^{III}.$$

(Recall that  $\mathcal{F}^{II}$  is the union of the Type II boundary components of  $\mathcal{F}^*$ , and  $\mathcal{F}^{III}$  is the (unique) Type III boundary component.)

First we prove a result on the period map  $\tilde{\mathfrak{p}}: |\mathcal{O}_{\mathbb{P}^3}(4)| \dashrightarrow \mathcal{F}^*$ . We define in the obvious way the subsets  $|\mathcal{O}_{\mathbb{P}^3}(4)|^I, |\mathcal{O}_{\mathbb{P}^3}(4)|^{II}, |\mathcal{O}_{\mathbb{P}^3}(4)|^{III}, |\mathcal{O}_{\mathbb{P}^3}(4)|^{IV} \subset |\mathcal{O}_{\mathbb{P}^3}(4)|$ .

**Lemma 2.18.** *The period map  $\tilde{\mathfrak{p}}$  is regular away from  $|\mathcal{O}_{\mathbb{P}^3}(4)|^{IV}$ , and*

$$(2.3) \quad \tilde{\mathfrak{p}}(|\mathcal{O}_{\mathbb{P}^3}(4)|^I) \subset \mathcal{F}, \quad \tilde{\mathfrak{p}}(|\mathcal{O}_{\mathbb{P}^3}(4)|^{II}) \subset \mathcal{F}^{II}, \quad \tilde{\mathfrak{p}}(|\mathcal{O}_{\mathbb{P}^3}(4)|^{III}) \subset \mathcal{F}^{III}.$$



*Proof.* Let  $X_0 \in (|\mathcal{O}_{\mathbb{P}^3}(4)| \setminus |\mathcal{O}_{\mathbb{P}^3}(4)|^{IV})$  be a quartic surface. Suppose that  $f: (B, 0) \rightarrow (|\mathcal{O}_{\mathbb{P}^3}(4)|, X_0)$  is a map from a smooth pointed curve, and that  $f(B \setminus \{0\})$  is contained in the locus of *smooth* quartics. Let  $p_f^0: (B \setminus \{0\}) \rightarrow \mathcal{F}^*$  be the composition  $\tilde{\mathfrak{p}} \circ (f|_{B \setminus \{0\}})$ , and let  $p_f: B \rightarrow \mathcal{F}^*$  be the extension to  $B$ . Then  $p_f(0)$  is independent of  $f$ . In fact, this follows from **Theorem 2.9**. In addition, we see that

- (1) if  $X_0 \in |\mathcal{O}_{\mathbb{P}^3}(4)|^I$ , then  $p_f(0) \in \mathcal{F}$ ,
- (2) if  $X_0 \in |\mathcal{O}_{\mathbb{P}^3}(4)|^{II}$ , then  $p_f(0) \in \mathcal{F}^{II}$ ,
- (3) and if  $X_0 \in |\mathcal{O}_{\mathbb{P}^3}(4)|^{III}$ , then  $p_f(0) \in \mathcal{F}^{III}$ .

Now assume that  $X_0 \in (|\mathcal{O}_{\mathbb{P}^3}(4)| \setminus |\mathcal{O}_{\mathbb{P}^3}(4)|^{IV})$ , and that  $X_0$  is in the indeterminacy locus of  $\tilde{\mathfrak{p}}$ . Then, since  $|\mathcal{O}_{\mathbb{P}^3}(4)|$  is smooth (normality would suffice), there exist smooth pointed curves  $(B_i, 0_i)$  for  $i = 1, 2$ , and maps  $f_i: (B_i, 0) \rightarrow (|\mathcal{O}_{\mathbb{P}^3}(4)|, X_0)$  such that  $f(B_i \setminus \{0_i\})$  is contained in the locus of *smooth* quartics, and the points  $p_{f_i}(0_i)$  (defined as above) are different, contradicting what was just stated. This proves that  $\tilde{\mathfrak{p}}$  is regular away from  $|\mathcal{O}_{\mathbb{P}^3}(4)|^{IV}$ . Equation (2.3) follows from Items (1), (2), (3) above.  $\square$

*Proof of Proposition 2.17.* First we notice that  $|\mathcal{O}_{\mathbb{P}^3}(4)|^I, |\mathcal{O}_{\mathbb{P}^3}(4)|^{II}, |\mathcal{O}_{\mathbb{P}^3}(4)|^{III}, |\mathcal{O}_{\mathbb{P}^3}(4)|^{IV}$  define a stratification of  $|\mathcal{O}_{\mathbb{P}^3}(4)|$ , because  $\mathcal{F}, \mathcal{F}^{II}, \mathcal{F}^{III}$  define a stratification of  $\mathcal{F}^*$ . Let  $|\mathcal{O}_{\mathbb{P}^3}(4)|^{ss} \subset |\mathcal{O}_{\mathbb{P}^3}(4)|$  be the open subset of GIT semistable quartics, and let  $\pi: |\mathcal{O}_{\mathbb{P}^3}(4)|^{ss} \rightarrow \mathfrak{M}$  be the quotient map. By definition (and the remark about  $|\mathcal{O}_{\mathbb{P}^3}(4)|^I, \dots, |\mathcal{O}_{\mathbb{P}^3}(4)|^{IV}$  defining a stratification of  $|\mathcal{O}_{\mathbb{P}^3}(4)|$ )  $\pi^{-1}(\mathfrak{M} \setminus \mathfrak{M}^{IV}) \subset (|\mathcal{O}_{\mathbb{P}^3}(4)| \setminus |\mathcal{O}_{\mathbb{P}^3}(4)|^{IV})$ . Hence  $\mathfrak{p}$  is regular away from  $\mathfrak{M}^{IV}$  because of **Lemma 2.18**. Lastly,  $\mathfrak{M}^I, \mathfrak{M}^{II}, \mathfrak{M}^{III}, \mathfrak{M}^{IV}$  define a stratification of  $\mathfrak{M}$  because  $\mathcal{F}, \mathcal{F}^{II}, \mathcal{F}^{III}$  define a stratification of  $\mathcal{F}^*$ .  $\square$

### 3. SHAH'S STRATIFICATION OF $\mathfrak{M}$ AND $\mathfrak{M}_h$

In the previous section, we have discussed the definition of a stratification by Hodge theoretic Type for  $\mathfrak{M}$  and  $\mathfrak{M}_h$ , see (2.1) and (2.2). In this section, we briefly review Shah's description of these stratifications ([Sha81, Theorem 2.4]). Then, we slightly refine these stratifications so that they match the stratifications of the relevant Baily-Borel compactifications defined in [LO16]. Essentially, our Type IV strata (that we label  $IV(k)$  and  $IV_h(k)$  respectively where  $k$  is an integer denoting dimension) are defined so that they match the stratifications  $W_k$  and  $W_h(k)$  defined in (1.14). Our strata either coincide or are substrata of Shah's strata (and in many of the substrata cases, all we do is to separate Shah's stratum into connected components).

**3.1. Type II strata for  $\mathfrak{M}$ .** The period map will extend along the Type II strata on the GIT side, and it will map these strata to Type II strata of the Baily-Borel compactification  $\mathcal{F}^*$ . The matching of these Type II strata and the discrepancies of dimensions will be discussed in the following section. For now, we just note that Shah's identified 8 Type II strata and that what is characteristic for each of these cases is the presence of a “ $j$ -invariant”. More precisely, in each of the cases there is either a simple elliptic singularity (of type  $\tilde{E}_6, \tilde{E}_7$ , or  $\tilde{E}_8$ ), a rational elliptic curve in the singularity locus, or a rational curve with 4 pinch points in the singularity locus. Hodge theoretically, this corresponds to the definition of Type II locus  $\text{Gr}_1^W H^2(X_0) \neq 0$  (N.B. simple Hodge theoretic considerations also show that if there are multiple  $j$ -invariants associated to  $X_0$ , e.g. two simple elliptic singularities, they have to coincide).

**Proposition 3.1.** *The Type II GIT boundary  $\mathfrak{M}^{II}$  consists of 8 irreducible boundary components. We label these components by II(1)–II(8). Let  $X$  be a quartic surface with closed orbit corresponding to the generic point of a Type II components. Then,  $X$  has the following description:*

- II(1) (cf.  $S\text{-II}(B, i, \tilde{E}_8)$ , also the generic locus in  $\sigma_1$ ) –  $\text{Sing}(X)$  consists of two double points of type  $\tilde{E}_8$ .

- II(2) (cf.  $S-II(B,i, \tilde{E}_7)$ , also the generic locus in  $\sigma_2$ ) –  $\text{Sing}(X)$  consists of two double points of type  $\tilde{E}_7$  and some rational double points.
- II(3) (cf.  $S-II(B,ii)$ , also the generic locus in  $\sigma_3$ ) –  $\text{Sing}(X)$  consists of two skew lines, each of which is an ordinary nodal curve with four simple pinch points.
- II(4) (cf.  $S-II(B,iii)$ , also the generic locus in  $\sigma_4$ ) –  $X$  consists of a plane and a cone over a nonsingular cubic curve in the plane (triple point of type  $\tilde{E}_6$ ).
- II(5) (cf.  $S-II(A,i)$ ) –  $\text{Sing}(X)$  consists of a double point  $p$  of type  $\tilde{E}_8$  and some rational double points such that no line in  $X$  passes through  $p$ .
- II(6) (cf.  $S-II(A,ii, \deg 2)$ ) –  $\text{Sing}(X)$  consists of a smooth conic  $C$  and possibly some rational double points.  $C$  is an ordinary nodal curve with 4 pinch points.
- II(7) (cf.  $S-II(A,ii, \deg 3)$ ) –  $\text{Sing}(X)$  consists of a twisted cubic  $C$  and possibly some rational double points.  $C$  is an ordinary nodal curve with 4 pinch points.
- II(8) (cf.  $S-II(A,ii, \deg 4)$ ) –  $\text{Sing}(X)$  consists of elliptic normal curve of degree 4 and possibly some rational double points (equivalently  $X$  is the union of two quadric surfaces that meet transversally).

Furthermore, the cases II(5)–II(8) correspond to stable quartics, while the cases II(1)–II(4) to strictly semistable quartics with generic stabilizer the 1-PSs  $\lambda_1, \dots, \lambda_4$  respectively (N.B.  $\overline{II}(i) = \sigma_i$  cf. **Lemma 2.2**).

*Proof.* This is precisely Shah [Sha81, Thm. 2.4]. The corresponding case in Shah’s Theorem is labeled by S-II(A/B, Case). Some of Shah’s cases (e.g. Theorem 2.4 II.A.ii) has several geometric sub-cases that are labeled in an obvious way (e.g. S-II(A,ii, deg 3) corresponding to the case when  $\text{Sing}(X)$  is a twisted cubic).  $\square$

**3.2. Type III strata for  $\mathfrak{M}$ .** For completeness, we list the geometric stratification of the Type III locus  $\mathfrak{M}^{III}$  in the GIT compactification. On the Baily-Borel side, all this strata will map to the unique Type III boundary point.

**Proposition 3.2.** *The polystable quartics  $X$  parameterized by  $\mathfrak{M}^{III}$  are of one of the following type:*

- III(1) (cf.  $S-III(B,iii)$ , also case  $\zeta$ ) –  $X$  consists of four planes with normal crossings (the tetrahedron). This is a single point  $\zeta \in \mathfrak{M}$  (cf. 2.3 (i)).
- III(2) (cf.  $S-III(B,ii, 4 \text{ lines})$ , also generic locus in  $\tau$ ) –  $X$  consists of two, nonsingular, quadric surfaces which intersect in a reduced curve  $C$  which consists of four lines such that its singularities consist of 4 double points. This gives a curve  $\tau^\circ \subset \mathfrak{M}$  (cf. 2.3 (ii)), where  $\tau^\circ = \tau \setminus \{\omega, \zeta\}$ .
- III(3) (cf.  $S-III(B,ii, 2 \text{ lines})$ ) –  $X$  consists of two, nonsingular, quadric surfaces which intersect in a reduced curve,  $C$ , of arithmetic genus 1.  $C$  consists of two lines such that its singularities consist of 2 double points; the dual graph of  $C$  is homeomorphic to a circle. This case is a specialization of the case II(8) above. Stabilizer  $\lambda_4 = (1, 0, 0, -1)$ .
- III(4) (cf.  $S-III(B,i, \deg 3)$ ) –  $\text{Sing}(X)$  consists of a nonsingular, rational curve of degree 3, and some rational double points.  $C$  is a strictly quasi-ordinary, nodal curve and the set of pinch points consists of two double pinch points. Each double pinch point lies on a line in  $X$ . Stabilizer  $\lambda_3 = (3, 1, -1, -3)$ . Also a specialization of the case II(7).
- III(5) (cf.  $S-III(B,i, \deg 2)$ ) –  $\text{Sing}(X)$  consists of a nonsingular, rational curve of degree 2, and some rational double points.  $C$  is a strictly quasi-ordinary, nodal curve and the set of pinch points consists of two double pinch points. Each double pinch point lies on a line in  $X$ . Stabilized by  $\lambda_4 = (1, 0, 0, -1)$ . Specialization of the case II(6).
- III(6) (cf.  $S-III(A,ii)$ ) –  $\text{Sing}(X)$  consists of a strictly quasi-ordinary nodal curve,  $C$ , and some rational double points such that no line in  $X$  passes through a double pinch point.  $C$  is a

nonsingular, rational curve of degree 2.  $X$  has either two double pinch points on  $C$  or one double pinch point and two simple pinch points on  $C$ . Specialization of the case II(6).

III(7) (cf.  $S\text{-III}(A,i)$ ) –  $\text{Sing}(X)$  consists of a double point,  $p$ , of type  $T_{2,3,r}$  and some rational double points such that no line in  $X$  passes through  $p$ . Specialization of the case II(5).

The cases III(1)–III(5) are strictly semistable. While the cases III(6) and III(7) are stable.

**3.3. Type IV strata for  $\mathfrak{M}$ .** As previously discussed, the period map is regular along the the Type I, II, and III locus. Thus, from the perspective of understanding the decomposition of the map

$$\mathfrak{M} \dashrightarrow \mathcal{F}^*$$

into simple birational maps, the relevant locus is the Type IV locus (the complement of the union of Type I, II, III loci). This locus is naturally stratified by the geometry of the singular locus. The following is just a slight refinement of Shah [Sha81, Theorem 2.4]:

**Proposition 3.3.** *The Type IV locus  $\mathfrak{M}^{IV}$  decomposes in the following strata:*

- IV(0a) (cf.  $S\text{-IV}(B,iii)$ ) –  $X$  consists of a non-singular quadric surface with multiplicity 2. (Case 2.3(iii)). The point  $\omega \in \mathfrak{M}$  corresponding to (generic) hyperelliptic quartics.
- IV(0b) (cf.  $S\text{-IV}(B,i, \deg 3)$ ) –  $\text{Sing}(X)$  consists of a nonsingular, rational curve,  $C$ , of degree 3;  $C$  is a simple cuspidal curve. The normalization of  $X$  is nonsingular. This is the tangent developable to the twisted cubic (Case 2.3(iv)). The corresponding point  $v \in \mathfrak{M}$  corresponds to unigonal K3s.
- IV(1) (cf.  $S\text{-IV}(B,ii)$ ) –  $X$  consists of two quadric surfaces,  $V_1, V_2$  tangent along a nonsingular conic  $C$  such that  $V_1 \cap V_2 = 2C$ . (Case 2.3(iii)). It corresponds to a curve inside  $\mathfrak{M}$ .
- IV(2) (cf.  $S\text{-IV}(B,i, \deg 2)$ ) –  $\text{Sing}(X)$  consists of a nonsingular, rational curve,  $C$ , of degree 2;  $C$  is a simple cuspidal curve. The normalization of  $X$  has exactly two rational double points. Stabilized by  $\lambda_4 = (1, 0, 0, -1)$
- IV(3)  $\text{Sing}(X)$  consists of a nodal curve,  $C$ , and rational double points such that no line in  $X$  passes through a non-simple pinch point.  $C$  is a nonsingular, rational curve of degree 2. Every point of  $X$  on  $C$  is a double point and the set of pinch points consists of a point of type  $E_{4,\infty}$ .
- IV(4)  $\text{Sing}(X)$  consists of a nodal curve,  $C$ , and rational double points such that no line in  $X$  passes through a non-simple pinch point.  $C$  is a nonsingular, rational curve of degree 2. Every point of  $X$  on  $C$  is a double point and the set of pinch points consists of either a point of type  $E_{3,\infty}$  and a simple pinch point or a point of type  $E_{4,\infty}$ .
- IV(5)  $\text{Sing}(X)$  consists of a double point,  $p$ , of type  $E_{3,r}$  and some RDPs such that no line in  $X$  passes through  $p$ . This case is a specialization of Case III(7) (and then II(8)).
- IV(6) (cf.  $S\text{-IV}(A,i, E_{14})$ ) –  $\text{Sing}(X)$  consists of a double point of type  $E_{14}$ .
- IV(7) (cf.  $S\text{-IV}(A,i, E_{13})$ ) –  $\text{Sing}(X)$  consists of a double point of type  $E_{13}$ .
- IV(8) (cf.  $S\text{-IV}(A,i, E_{12})$ ) –  $\text{Sing}(X)$  consists of a double point of type  $E_{12}$ .

*Remark 3.4.* Note that there are natural inclusions  $\text{IV}(k) \subset \overline{\text{IV}(k+1)}$  with the exception  $k = 4$  (N.B.  $\text{IV}(4) \subset \overline{\text{IV}(6)}$ ). For instance, we have the following adjacencies for the exceptional unimodal singularities (aka Dolgachev singularities):  $E_{14} \longrightarrow E_{13} \longrightarrow E_{12}$  (see [AGLV98, p. 159]).

**Definition 3.5.** We define

$$W_k = \overline{\text{IV}(k)},$$

with the following two exceptions:  $W_0 = \overline{\text{IV}(0a)}$ , and we skip the case  $k = 5$ .

*Remark 3.6.* For quartics singular along a twisted cubic, we have the inclusions:

$$\text{IV}(0b) \subset \text{III}(4) \subset \text{II}(7).$$

*Remark 3.7.* Clearly, II(1), III(1), and IV(1) form a single stratum. The degeneracy condition is that there is a line passing through  $p$ , cf. [Sha81, Cor. 2.3 (i)]: *an isolated, non rational, double point of Type 1 through which passes a line contained in  $X$ .*

*Remark 3.8.* Cases II(5) and its specializations III(7) and IV(6–8) were studied by Urabe [Ura84].

We summarize our predictions on the correspondence of loci in  $\mathcal{F}^*$  and those in  $\mathfrak{M}$  in the 1 below.

Codim	Critical $\beta$	BB locus $Z(\beta)$	GIT locus $W(\beta)$ ( $\dim W = \text{codim } Z - 1$ )
1	1	$H_h$	IV(0a): double quadric
1	1	$H_u$	IV(0b): developable quartic
2	$\frac{1}{2}$	$\Delta^{(2)}$	IV(1): 2 quadrics meeting in a conic
3	$\frac{1}{3}$	$\Delta^{(3)}$	IV(2): double conic, cuspidal type
4	$\frac{1}{4}$	$\Delta^{(4)}$	IV(3): $E_{4,\infty}$ -locus
5	$\frac{1}{5}$	$\Delta^{(5)}$	IV(4): $E_{3,0}$
6	$\frac{1}{5}$	$\Delta^{(6)}$	IV(5): $E_{3,\infty}$ and $E_{3,r}$
7	$\frac{1}{6}$	unigonal in $\Delta^{(6)}$ ( $T_{3,3,4}$ -polarized K3)	IV(6): $E_{14}$ -locus
8	$\frac{1}{7}$	unigonal in $\Delta^{(7)}$ ( $T_{2,4,5}$ -polarized K3)	IV(6): $E_{13}$ -locus
9	$\frac{1}{9}$	unigonal in $\Delta^{(8)}$ ( $T_{2,3,7}$ -polarized K3)	IV(8): $E_{12}$ -locus

TABLE 1. The geometry of the variation of models  $\mathcal{F}(\beta)$

*Remark 3.9.* We recall that the  $Z_9$  locus (described above as the unigonal divisor inside  $\Delta^{(8)} \cong \mathcal{F}(11)$ ) can be also described as one of the two components of  $\Delta^{(9)}$ . With this description, the jump from  $\frac{1}{7}$  to  $\frac{1}{9}$  is less surprising: the critical  $\beta = \frac{1}{9}$  comes from having 9 independent sheets of  $\Delta$  meeting along the  $Z^9$  locus.

The matching of  $H_h$  and  $H_u$  with IV(0a) and IV(0b) is quite clear geometrically, and discussed in Shah [Sha81] Sections 4 and 3 respectively. We revisit this in **Subsection 5.2** and **Subsection 5.1** respectively. The claimed matching for  $\beta = \frac{1}{2}$  is discussed in **Subsection 5.4**. Finally, in **Subsection 5.5** we give some evidence for the matching corresponding to the case  $\beta \in \{\frac{1}{6}, \frac{1}{7}, \frac{1}{9}\}$ . We don't say much about the remaining cases, we conjecture the matching based on obvious analogies (and some evidence is easily available). In the case of hyperelliptic quartics, stronger evidence for the matching of Table 1 comes from GIT (compare with Table 6). We discuss this in **Section 6**.

*Remark 3.10.* While the entire framework of the paper is similar to the Hassett-Keel program for curves. We note that the geometric analogy with the HK is striking in the case of flips occurring for  $\beta \in \{\frac{1}{6}, \frac{1}{7}, \frac{1}{9}\}$ . Namely, to pass from the  $E_l$  ( $l = 12, 13, 14$ ) locus on the GIT side to the periods side, one needs to perform a KSBA semistable replacement. This is completely analogous to the stable reduction for cuspidal curves, which leads to the elliptic tail replacement (or globally to the first birational modification:  $\mathfrak{M}_g \rightarrow \mathfrak{M}_g^{ps} \cong \mathfrak{M}_g(\frac{9}{11})$ ). This part is closely related to the work of Hassett [Has00] (stable replacement for curves). This is expanded on in **Subsection 5.5**.

*Remark 3.11.* It should be possible to recover the critical  $\beta$  by using the techniques of Alper-Fedorchuck-Smyth [AFS10], i.e. starting from the GIT side and identifying double covers of quadrics with  $\mathbb{C}^*$  stabilizer. This is related (but different) to our approach from **Section 6**.

**3.4. Hodge-theoretic stratification of  $\mathfrak{M}_h$ .** A completely analogous description of the stratification in Types applies to  $\mathfrak{M}_h$ , the GIT quotient for  $(4, 4)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$ . As already discussed, the Type refers to the singularities of the double cover, while when we state the stratification we refer to branch curve  $C$  (e.g. assume  $C$  is a curve on a quadric (smooth or cone) with an  $\tilde{E}_8$  singularity, and possibly ADE singularities, then we say  $C$  is of Type II since the associated double cover  $X$  is a Type II degeneration of  $K3$ s). Also, as discussed in **Subsection 2.3**, the stratification on  $\mathfrak{M}_h$  is induced from that on  $\mathfrak{M}$  (via the blow-up of the point  $\omega$  (or  $IV(0a)$ )). In conclusion, we have a parallel statement to **Proposition 3.3**. The main difference is that there is no analogue of  $IV(0b)$  (or  $v$ ) stratum, and that the dimensions decrease by 1. The precise statement is given in Shah [Sha81, Theorem 4.8].

**Proposition 3.12.** *The Type IV locus  $\mathfrak{M}_h^{IV}$  decomposes in the following strata:*

- $IV_h(0)$  (cf.  $S-IV_h(B, ii)$ ) –  $C$  is a conic with multiplicity 4.
- $IV_h(1)$  (cf.  $S-IV_h(B, i)$ ) –  $C$  consists of a triple conic plus another conic.
- $IV_h(2)$   $C$  contains a double conic and a single pinch point of type  $E_{4,\infty}$ .
- $IV_h(3)$   $C$  has a singularity of type  $E_{3,0}$ .
- $IV_h(4)$   $C$  has a singularity of type  $E_{3,r}$  (with  $r > 0$  and possibly  $r = \infty$ ).
- $IV_h(5)$  (cf.  $S-IV_h(A, i, E_{14})$ ) –  $C$  has a singularity of type  $E_{14}$ .
- $IV_h(6)$  (cf.  $S-IV_h(A, i, E_{13})$ ) –  $C$  has a singularity of type  $E_{13}$ .
- $IV_h(7)$  (cf.  $S-IV_h(A, i, E_{12})$ ) –  $C$  has a singularity of type  $E_{12}$ .

*Remark 3.13.* The cases of  $E_{3,\infty}$  and  $E_{4,\infty}$  correspond to a situation when there is a double conic (simply because there are non-isolated singularities for a  $(4, 4)$  curve. Thus, there is a non-reduced component. A simple analysis reduces to the case of type  $(1, 1)$ ). The residual  $(2, 2)$  curve will cut the conic in 4 points. Typically, the 4 points will be distinct giving 4 ordinary pinch points. If two of the pinch points come together, one has Type III singularities. The cases when 3 or 4 coincide correspond to  $E_{3,\infty}$  and  $E_{4,\infty}$  respectively. The fact that  $E_{3,r}$  ( $r > 0$ ) is put together with  $E_{3,\infty}$  should be understood that they occur via perturbation of a non-isolated singularity of type  $E_{3,\infty}$ .

**Definition 3.14.** We define

$$W_{h,k} = \overline{IV_h(k)},$$

(skipping the  $k = 4$ ). (Note that inclusions similar to those of **Remark 3.4** hold among the Type  $IV_h$  loci, and thus we have the expected inclusions for  $W_{h,k}$ .)

#### 4. LOOIJENGA'S $\mathbb{Q}$ -FACTORIALIZATION

The predictions of our previous paper [LO16] are concerned with the birational transformations that occur in the interior of the period domain  $\mathcal{F} = \mathcal{D}/\Gamma$ . Our working assumption is that all the modifications that occur at the boundary of the Baily-Borel compactification  $\mathcal{F}^*$  are in fact explained by Looijenga's work [Loo03b]. Namely, as already explained in the earlier sections, the first modification of  $\mathcal{F}^* = \mathcal{F}(0)$  occurs for  $0 < \epsilon \ll 1$ :  $\widehat{\mathcal{F}} = \mathcal{F}(\epsilon)$  is the  $\mathbb{Q}$ -factorialization of the boundary divisor  $\Delta = \frac{1}{2}(H_h + H_u)$  (i.e.  $\widehat{\mathcal{F}} \rightarrow \mathcal{F}^*$  is a small birational map, that leaves the interior  $\mathcal{F} = \mathcal{D}/\Gamma$  unchanged, and makes the transform of  $\Delta$  a  $\mathbb{Q}$ -Cartier divisor). Our expectation is that the center of the birational transformations for  $\mathcal{F}(\beta)$  for  $\beta \in (\epsilon, 1]$  will not be contained in the boundary  $\widehat{\mathcal{F}} \setminus \mathcal{F}$ . The purpose of this section is to give evidence towards this claim.



Specifically, Looijenga [Loo03b] has constructed the  $\mathbb{Q}$ -factorialization  $\widehat{\mathcal{F}}$  as a semi-toric compactification which is determined in a combinatorial way by the arithmetic hyperplane (or pre-Heegner divisor in the language of [LO16])  $\mathcal{H} = \pi^{-1}(H_h \cup H_u)$  (where  $\pi : \mathcal{D} \rightarrow \mathcal{D}/\Gamma$  is the natural projection). The stratification of the Baily-Borel compactification induces (by pull-back) a stratification of the boundary of  $\widehat{\mathcal{F}}$  into 9 Type II strata, and 1 Type III stratum (of course some of these strata might become reducible in  $\widehat{\mathcal{F}}$ ). The expectation stated in the previous paragraph implies that these boundary strata should be equivalent birationally with the Type II and Type III GIT strata described in **Section 3**. The structure of the Type III stratum in  $\widehat{\mathcal{F}}$  is quite complicated (it is here that the toric part of the compactification comes into play), and we don't have anything to say here. What we do instead is to match, via a geometric heuristic (discovered by Friedman [Fri84]), the Type II boundary components in  $\mathfrak{M}$  and  $\widehat{\mathcal{F}}$ , and to check that their dimensions match. With a bit more effort, it might be possible to check that there exists an extended period map  $\mathfrak{M} \dashrightarrow \widehat{\mathcal{F}}$  which is regular along the Type II boundary in  $\mathfrak{M}$  and its restrictions give birational isomorphisms between the Type II components in  $\mathfrak{M}$  and  $\widehat{\mathcal{F}}$ .

Before we proceed with our computations, we note that there is a glaring discrepancy that seems to be against our predictions above: there are 8 Type II components in  $\mathfrak{M}$ , while there are 9 in  $\widehat{\mathcal{F}}$ . This is in fact not a contradiction, as we will see that the missing component is contained in the closure of one of the  $Z^k \subset \mathcal{F}$  strata and thus will disappear in the associated flip (and it will be hidden in the Type IV locus in  $\mathfrak{M}$ ). We note that compared with the case of degree 2  $K3$  surfaces ([Sha80], [Loo86]) or cubic fourfolds ([Laz10], [Loo09]), this is a new phenomenon which points to the interesting nature of the quartic example.

**4.1. The Baily-Borel and Looijenga compactifications for quartic surfaces.** We start by recalling that the locally symmetric variety  $\mathcal{F} = \mathcal{D}/\Gamma$  has a canonical compactification  $\mathcal{F}^*$  which is obtained by adding curves (in fact modular curves), the Type II boundary components, and some points, the Type III boundary components. The structure of the Baily-Borel compactification for quartic surfaces was worked out by Scattone [Sca87]: there are 9 Type II boundary components, and a single Type III boundary component. In the appendix **Section A**, we review the Baily-Borel compactification for the  $D$ -tower  $\mathcal{F}(N)$ , in particular for the moduli space of quartic  $K3$ s  $\mathcal{F}$  (case  $N = 19$ ) and hyperelliptic quartic surfaces  $\mathcal{F}_h$  (i.e.  $N = 18$ ). For the purpose of this section, we only recall:

**Proposition 4.1** (Scattone [Sca87]). *The boundary of the Baily-Borel compactification  $\mathcal{F}^*$  of the moduli space of quartic surfaces consists of 9 Type II boundary components, and a single Type III component. The Type II boundary components are naturally labeled by a rank 17 negative definite lattice as follows:  $D_{17}$ ,  $D_9 \oplus E_8$ ,  $D_{12} \oplus D_5$ ,  $D_3 \oplus (E_7)^2$ ,  $A_{15} \oplus D_2$ ,  $A_{11} \oplus E_6$ ,  $(D_8)^2 \oplus D_1$ ,  $D_{16} \oplus D_1$ , and  $(E_8)^2 \oplus D_1$  respectively (see **Section A** for details).*

The locally symmetric variety  $\mathcal{F}$  has at worst finite quotient singularity, and thus it is  $\mathbb{Q}$ -factorial. Since  $\mathcal{F} \subset \mathcal{F}^*$  is a high codimension compactification, any divisor of  $\mathcal{F}$  extends as a Weil divisor, but typically not a  $\mathbb{Q}$ -Cartier divisor. Looijenga [Loo03b] has constructed a  $\mathbb{Q}$ -factorialization associated to any arithmetic hyperplane arrangement (or equivalently pre-Heegner divisor in the terminology of [LO16]). Here, we are interested in the  $\mathbb{Q}$ -factorialization of the closure of  $\Delta = \frac{1}{2}(H_u + H_h)$ .

**Definition 4.2.** Let  $\widehat{\mathcal{F}} \rightarrow \mathcal{F}^*$  be the Looijenga  $\mathbb{Q}$ -factorialization associated to the hyperplane arrangement  $\mathcal{H} = \pi^{-1}(H_h \cup H_u)$  (where  $\pi : \mathcal{D} \rightarrow \mathcal{D}/\Gamma$  is the natural projection) of hyperelliptic and unigonal pre-Heegner divisors.

From our perspective, it is immediate to see that the  $\mathbb{Q}$ -factorialization coincides with one of our models:

**Proposition 4.3.** *With notation as above. The following holds*

$$\widehat{\mathcal{F}} \cong \mathcal{F}(\epsilon)$$

for  $0 < \epsilon \ll 1$ .

*Proof.* By construction,  $\widehat{\mathcal{F}}$  has the property that  $\lambda + \epsilon\Delta$  is  $\mathbb{Q}$ -Cartier and ample (N.B. the relative ampleness of  $\Delta$  is not explicitly stated in [Loo03b], but this is precisely what Looijenga checks). Additionally  $\widehat{\mathcal{F}}$  differs in codimension 2 from  $\mathcal{F}$ . Thus

$$\mathrm{Proj} R(\widehat{\mathcal{F}}, \lambda + \epsilon\Delta) \cong \mathrm{Proj}(\mathcal{F}, \lambda + \epsilon\Delta) = \mathcal{F}(\epsilon).$$

□

*Remark 4.4.* According to the discussion of [KM98, Ch. 6], the  $\mathbb{Q}$ -factorialization of  $\Delta$  is unique: it is either  $\mathcal{F}(\epsilon)$  or  $\mathcal{F}(-\epsilon)$  (depending on the requested relative ampleness). The main issue is that the finite generation of the ring section defining  $\mathcal{F}(\epsilon)$  is not a priori guaranteed. Looijenga [Loo03b] makes use of the special structure of the Baily-Borel compactification (e.g. the tube domain structure near the boundary, and the existence of toroidal compactifications) to obtain that the  $\mathbb{Q}$ -factorialization is well defined, and furthermore to get an explicit description of it.

*Remark 4.5.* In fact, the predictions of [LO16] (see esp. Proposition 5.4.5) give precise bounds for  $\epsilon$ : in the quartic case, the above proposition should hold for  $0 < \epsilon < \frac{1}{9}$ .

The main interest for us about the compactification  $\widehat{\mathcal{F}}$  is the dimension of the Type II boundary strata.

**Proposition 4.6.** *The dimensions of the Type II strata in the compactification  $\widehat{\mathcal{F}}$  are given in Table 2.*

$D_{17}$	1	$D_9 \oplus E_8$	10	$D_{12} \oplus D_5$	6
$D_3 \oplus (E_7)^2$	4	$A_{15} \oplus D_2$	3	$A_{11} \oplus E_6$	1
$(D_8)^2 \oplus D_1$	2	$D_{16} \oplus D_1$	6	$(E_8)^2 \oplus D_1$	2

TABLE 2. Dimension of the boundary strata in  $\widehat{\mathcal{F}}$

*Proof.* A type II boundary component is determined by the choice an isotropic rank 2 primitive sublattice  $E \subset \Lambda (= \Lambda_{19} = E_8^2 \oplus U \oplus \langle -4 \rangle)$  (up to the action of the monodromy group). The label associated to a Type II boundary component is the root sublattice contained in the negative definite rank 17 lattice  $E_\Lambda^\perp/E$  (with the convention of including also  $D_1 = \langle -4 \rangle$  in the root lattice). According to [Sca87], this is a complete invariant for a Type II boundary component in the case of quartic surfaces.

The construction of Looijenga [Loo03b] (see esp. Section 3 and Proposition 3.3 of loc. cit.) depends on the linear space

$$L := \left( \bigcap_{H \in \mathcal{H}, E \subset H} (H \cap E^\perp) \right) / E \subset E^\perp / E.$$

More precisely, let  $M := E^\perp / E$ ;  $M$  is a negative definite rank 17 lattice. Then, we recall that the fiber over a point  $j$  in the type II boundary component (recall each Type II boundary component is a modular curve, here  $\mathfrak{h}/\mathrm{SL}(2, \mathbb{Z})$ ) associated to  $E$  is simply the quotient of the abelian variety  $J(\mathcal{E}_j) \otimes_{\mathbb{Z}} M$  by a finite group (here  $\mathcal{E}_j$  denotes the elliptic curve of modulus  $j$ , and  $J(\mathcal{E}_j)$  its Jacobian). What Looijenga has observed is that  $L$  is the null-space of the restriction to the toroidal boundary

(of Type II) of the linear system determined by the hyperplane arrangement  $\mathcal{H}$ . And thus, the fiber for the  $\mathbb{Q}$ -factorialization (which as discussed above corresponds to the Proj of the ring of sections of  $\lambda + \epsilon\Delta$ ; also recall (the pull-back of)  $\lambda$  restricts to trivial on the toroidal boundary) over the point  $j$  in the Type II boundary component associated to  $E$  is (up to finite quotient)  $J(\mathcal{E}_j) \otimes_{\mathbb{Z}} M/L$ .

Now, we recall that the lattice  $\Lambda$  can be primitively embedded into the Borcherds lattice  $II_{2,26}$  with orthogonal complement  $D_7$  (in a unique way). We fix

$$\Lambda \hookrightarrow II_{2,26}$$

and  $R = \Lambda^\perp \cong D_7$ . With respect to this embedding, a hyperelliptic hyperplane corresponds to an extension of  $R$  to a (primitively embedded)  $D_8$  into  $II_{2,26}$ , while a unigonal divisor to a  $E_8$ . Successive intersections of hyperplanes from  $\mathcal{H}$  correspond to extensions of  $R = (D_7)$  into  $D_k$  lattices. Similarly, if  $E$  is rank 2 isotropic (primitively embedded), then we recall that  $M = E^\perp/E$  can be embedded into one of the 24 Niemeier lattices (i.e. rank 24 negative definite even unimodular lattices) with orthogonal complement  $D_7$ . The same considerations as before apply: a hyperelliptic divisor correspond to an extension to  $D_8$  (and repeated intersections to  $D_{7+k}$ ), while a unigonal one corresponds to an extension to  $E_8$ . In table 8, we discuss the possible embeddings of  $D_k$  lattices into Niemeier lattices (the relevant cases are those containing  $D_7$ , and then the computation of  $L$  corresponds to finding the maximal  $D_k$  extending this  $D_7$ ). By inspection, one obtains the dimensions claimed in Table 2. The only exception is the case  $D_{17}$  (in which case  $D_7$  extends to  $D_{24}$  – see Table 8) for which  $L = 0 \subset M$ , and thus the Heegner divisor is already  $\mathbb{Q}$ -Cartier (and no modification is necessary; see [Loo03b, Cor. 3.5]).  $\square$

**4.2. Matching Type II boundary strata.** To understand the matching of the GIT and BB Type II strata, one needs to consider a generic smoothing of a Type II surface  $X_0$  and after performing a semistable reduction to compute the limit MHS with  $\mathbb{Z}$ -coefficients. The case of degree 2 K3 surfaces was analyzed by Friedman [Fri84]. Our situation is quite similar, and we have the following heuristic (see esp. [Fri84, Rem. 5.6]) to attach to a GIT Type II boundary a BB Type II boundary:

**Heuristic 4.7.** *For the Type II case there is always a well defined  $j$ -invariant (recall  $h^{1,0} = 1$  for the associated MHS in the Type II case). And we label the component as follows depending on the singularities of  $X$ :*

- $\tilde{E}_r$  by  $E_r$
- rational normal curve  $C$  of degree  $d$  by  $D_{4d+4}$
- elliptic curve  $C$  of degree  $d$  by  $A_{4d-1}$ .

( $4d$  is the number of special points on  $C$  in a pencil degeneration). Additionally, we should add also the subroot lattice from  $K_{\tilde{X}}^\perp \subset \text{Pic}(\tilde{X})$  where  $\tilde{X}$  is the resolution of  $X$  (e.g. if  $\tilde{X}$  is a degree 2 del Pezzo, we add  $E_7$ ).

With this Heuristic, we can give a match of the Type II boundary on the GIT and Looijenga compactification side. Furthermore, we note that dimensions match.

**Proposition 4.8.** *The extended period map maps birationally the 8 Type II boundary components of the GIT quotient  $\mathfrak{M}$  (cf. **Proposition 3.1**) to 8 of the Type II boundary components of the Looijenga  $\mathbb{Q}$ -factorialization  $\widehat{\mathcal{F}}$  as given in Table 3.*

*Proof.* We illustrate the computation of dimensions only for the highest dimensional case: II(5), i.e. quartics that have a single  $\tilde{E}_8$  singularity. From Shah [Sha81], a generic surface  $X_0$  of this type is GIT stable. On the other hand, the results of [dPW00] and [ST99] imply in particular (loc. cit. give general conditions in terms of total Tjurina number) that the universal family of quartic

surfaces versally unfolds the  $\tilde{E}_8$  singularity. From this two results, it follows that the codimension of the locus with a fixed  $\tilde{E}_8$  singularity is  $10 = \mu(\tilde{E}_8)$  (where  $\mu$  is the Milnor, and also, in this case, Tjurina number), but there is an additional 1-dimensional deformation corresponding to varying the modulus of the simple elliptic singularity. In conclusion, the II(5) locus has codimension 9 (or equivalently dimension 10) in the GIT quotient  $\mathfrak{M}$ .

The computations of dimensions of the stratum  $E_8 \oplus D_9$  was already done in **Proposition 4.6**. Here we point out that this case corresponds to  $D_{16} \oplus E_8$  in Table 8. In that situation, the maximally embedded  $D_l$  is  $D_{16}$  (see Table 8), which means (using the notation of **Proposition 4.6**)  $\dim M/L = 9$  (N.B.  $16 = 7 + 9$ ). Thus the fiber of  $\widehat{\mathcal{F}} \rightarrow \mathcal{F}^*$  over a point  $j$  in the Type II component labeled by  $E_8 \oplus D_9$  is 9. Then again, by varying  $j$ , we obtain a 10-dimensional component.

To match the Type II component II(5) on the GIT side with the component labeled  $E_8 \oplus D_9$  in  $\widehat{\mathcal{F}}$ , one notes that the semistable (or Kulikov) model in this situation is  $\tilde{X}_0 \cup_E T$ , where  $\tilde{X}_0$  is the resolution of the quartic surface with an  $\tilde{E}_8$  singularity,  $T$  is a “tail” (depending on the direction of the smoothing). In this situation, it is well known (and easily seen explicitly) that  $T$  is a degree 1 del Pezzo surface, whose primitive cohomology is  $E_8$ .  $\tilde{X}_0$  is a rational surface with primitive cohomology  $D_9$  (the full cohomology is  $I_{1,9}$  and the polarization is of degree 4). Finally, the gluing curve  $E$  is an elliptic curve (with self-intersection 1 on  $T$  and  $-1$  on  $\tilde{X}_0$ ), which gives the modulus  $j$  discussed above. The matching at the level of Baily-Borel compactification is analogous to the discussion of Friedman [Fri84] (briefly, the discrete part  $E_8 \oplus D_9$  corresponds the graded weight 2 piece of the mixed Hodge structure (which can be defined over  $\mathbb{Z}$  here), while the elliptic curve to the weight 1 piece). Finally, when passing to Looijenga  $\mathbb{Q}$ -factorialization, the fiber for fixed  $j$ -invariant is just  $J(\mathcal{E}_j) \otimes_{\mathbb{Z}} M/L$ . In this situation,  $M/L = D_9$ , which corresponds that only  $X_0$  varies in moduli at the level of  $\mathfrak{M}$  (the tail  $T$  only appears if one blow-ups the locus II(5) inside  $\mathfrak{M}$ , which roughly corresponds to a simultaneous semi-stable resolution for the  $\tilde{E}_8$  singularity). It is possible to see that the restriction of the extended period map

$$\mathfrak{M} \dashrightarrow \widehat{\mathcal{F}} \rightarrow \mathcal{F}^*$$

(which extends over the Type II and III locus) to the locus II(5) is nothing else but the period map for the anticanonical pair  $(\tilde{X}_0, E)$  (which is the same geometrically as the singular quartic  $X_0$ ) – see [GHK15] and [Fri13] for a discussion.  $\square$

*Remark 4.9* (Kulikov models). It is not hard to produce Kulikov models for each of the Type II degenerations above. For instance, in a semi-stable degeneration, each of the  $\tilde{E}_r$  singularities will be replaced by del Pezzo of degree  $(9-r)$ . As an example, the case II(1) corresponding to a quartic with 2  $\tilde{E}_8$  singularities will give 2 degree 1 del Pezzo surfaces, glued to a elliptic ruled surface (which is in the fact the resolution of the singular quartic; the del Pezzo surface are “tails” coming from the singularity; see §5.5.3 below for related computations). The case of quartics singular along a curve typically are obtained by projection from a rational surfaces (frequently del Pezzo). For instance the case II(6) is obtained by projecting a degree 4 del Pezzo from a point in  $\mathbb{P}^4$ . How about the associated label  $D_{12} \oplus D_5$  (whose Hodge theoretic meaning is the  $\mathbb{Z}$ -structure on the graded 2 piece of the limit MHS; see [Fri84] for discussion)?  $D_5 (= E_5)$  is the primitive cohomology of the associated degree 4 del Pezzo. On the other hand  $D_{12}$  is coming from the singularity of the quartic (in this case a conic) and the heuristic given above. The justification for this is explained in [Fri84, Sect. 5].

**4.3. The missing stratum.** The reader might be puzzled by the fact the BB stratum corresponding to  $D_{17}$  does not seem to occur on the list of Type II GIT boundary. We can explain this as follows. First of all as noted in **Proposition 4.6**, along this stratum the hyperelliptic divisor is

GIT stratum	BB stratum	Dimension
II(1)	$(E_8)^2 \oplus D_1$	2
II(2)	$(E_7)^2 \oplus A_3$	4
II(3)	$(D_8)^2 \oplus D_1$	2
II(4)	$E_6 \oplus A_{11}$	1
II(5)	$E_8 \oplus D_9$	10
II(6)	$D_{12} \oplus D_5$	6
II(7)	$D_{16} \oplus D_1$	6
II(8)	$A_{15} \oplus (A_1)^2$	3

TABLE 3. Matching of the Type II strata

$\mathbb{Q}$ -factorial and thus it is not affected by the  $\mathbb{Q}$ -factorialization. Then, this is precisely the boundary component that is contained in all elements of the  $D$ -tower. (this is the component that survives when we go to low dimensions, see **Section A**). As discussed the entire  $H_h^{(k)}$  (when we get to codimension 9) is contracted and then flipped (i.e. there is no deeper flip). This is indeed compatible with a theorem of Looijenga which identifies  $\mathcal{F}(10)^*$  with certain weighted projective space and with the moduli of  $T_{2,3,7}(= E_8 \oplus U)$  marked  $K3$ s (see **Subsection 5.5**). In conclusion, the 9th boundary component is all flipped at once together with a big stratum, and thus will not be visible individually. It will be hidden in the  $E_{12}$  stratum on the GIT side.

## 5. EVIDENCE FOR HKL PROGRAM FOR QUARTIC SURFACES

**5.1. Blow up of the point  $v$ .** The point  $v$  is an isolated point of the indeterminacy locus of the period map  $\mathbf{p}$  (see **Proposition 3.3**). The behavior of  $\mathbf{p}$  in a neighborhood of  $v$  is analogous to that of the period map of the moduli space of plane sextics in a neighborhood of the orbit of  $3C$  (see [Sha80], [Loo86], [Laz16, Thm. 1.9]), where  $C \subset \mathbb{P}^2$  is a smooth conic, and is treated in Section 3 of Shah [Sha81]. Shah's results imply that by blowing up a subscheme of  $\mathfrak{M}$  supported at  $v$ , one resolves the indeterminacy of  $\mathbf{p}$  in  $v$ ; the main result is stated in **Subsubsection 5.1.5**.

**5.1.1. The germ of  $\mathfrak{M}$  at  $v$  in the analytic topology.** We will apply Luna's étale slice Theorem in order to describe an analytic neighborhood of  $v$  in  $\mathfrak{M}$ . Let  $T \subset \mathbb{P}^3$  be the twisted cubic  $\{[\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3] \mid [\lambda, \mu] \in \mathbb{P}^1\}$ , and let  $X$  be the tangent developable of  $T$ , i.e. the union of lines tangent to  $T$ . A generator of the homogeneous ideal of  $X$  is given by

$$(5.1) \quad f := 4(x_1x_3 - x_2^2)(x_0x_2 - x_1^2) - (x_1x_2 - x_0x_3)^2.$$

Thus  $X$  is a polystable quartic representing the point  $v$ . The group  $\mathrm{PGL}(2)$  acts on  $T$  and hence on  $X$ ; it is clear that  $\mathrm{PGL}(2) = \mathrm{Aut}(X)$ . In order to describe an étale slice for the orbit  $\mathrm{PGL}(4)X$  at  $X$  we must decompose  $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$  into irreducible  $\mathrm{SL}_2$ -submodules. For  $d \in \mathbb{N}$ , let  $V(d)$  be the irreducible  $\mathrm{SL}_2$ -representation with highest weight  $d$  i.e.  $\mathrm{Sym}^d V(1)$  where  $V(1)$  is the standard 2-dimensional  $\mathrm{SL}_2$ -representation. A straightforward computation gives the decomposition

$$(5.2) \quad H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \cong V(0) \oplus V(4) \oplus V(6) \oplus V(8) \oplus V(12).$$



The trivial summand  $V(0)$  is spanned by  $f$ , and the projective tangent space at  $V(f)$  to the orbit  $\mathrm{PGL}(4)V(f) \subset |\mathcal{O}_{\mathbb{P}^3}(4)|$  is equal to  $\mathbb{P}(V(0) \oplus V(4) \oplus V(6))$ . We have a natural map

$$(5.3) \quad \begin{array}{ccc} V(8) \oplus V(12) // \mathrm{SL}(2) & \longrightarrow & \mathfrak{M}, \\ [g] & \mapsto & [V(f+g)] \end{array}$$

mapping  $[0]$  to  $v$ . By Luna's étale slice Theorem, the map is étale at  $[0]$ . In particular we have an isomorphism of analytic germs

$$(5.4) \quad (V(8) \oplus V(12) // \mathrm{SL}(2), [0]) \xrightarrow{\sim} (\mathfrak{M}, v).$$

5.1.2. *Moduli and periods of unigonal K3 surfaces.* Let

$$(5.5) \quad \Omega := S^\bullet(V(8)^\vee \oplus V(12)^\vee),$$

and define a grading of  $\Omega$  by declaring that  $V(8)^\vee$  is the summand of degree 2, and  $V(12)^\vee$  is the summand of degree 3. Then  $\mathrm{SL}(2)$  acts on  $\mathrm{Proj} \Omega$ , and  $\mathcal{O}_{\mathrm{Proj} \Omega}(1)$  is naturally linearized; thus there is a GIT quotient

$$(5.6) \quad \mathfrak{M}_u := \mathrm{Proj} \Omega // \mathrm{SL}(2).$$

Shah (see Theorem 4.3 in [Sha80]) proved that  $\mathfrak{M}_u$  is a compactification of the moduli space for unigonal K3 surfaces, i.e. there is an open dense subset  $\mathfrak{M}_u^I \subset \mathfrak{M}_u$  which is the moduli space for such K3's. Moreover, the period map is regular

$$(5.7) \quad \mathfrak{M}_u \xrightarrow{p_u} \mathcal{F}_{\mathrm{II}_{2,18}}(O^+(\mathrm{II}_{2,18}))^*,$$

and it defines an isomorphism  $\mathfrak{M}_u^I \xrightarrow{\sim} \mathcal{F}_{\mathrm{II}_{2,18}}(O^+(\mathrm{II}_{2,18}))$ . We recall that we have a natural regular map

$$(5.8) \quad \mathcal{F}_{\mathrm{II}_{2,18}}(O^+(\mathrm{II}_{2,18}))^* \longrightarrow \mathcal{F}^*,$$

whose restriction to  $\mathcal{F}_{\mathrm{II}_{2,18}}(O^+(\mathrm{II}_{2,18}))$  is an isomorphism onto the unigonal divisor  $H_u$ , see Subsection 1.5 of [LO16].

5.1.3. *Weighted blow-up.* We recall the construction of the weighted blow up in the case where the base is smooth (not a general cyclic quotient singularity). We refer to [KM92, And16] for details. Let  $(x_1, \dots, x_n)$  be the standard coordinates on  $\mathbb{A}^n$ . Let  $(a_1, \dots, a_n) \in \mathbb{N}_+^n$ , and let  $\sigma$  be the weight given by  $\sigma(x_i) = a_i$ . The weighted blow-up  $B_\sigma(\mathbb{A}^n)$  with weight  $\sigma$  is a toric variety defined as follows. Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $\mathbb{R}^n$ , and  $C \subset \mathbb{R}^n$  the convex spanned by  $e_1, \dots, e_n$ , i.e. the cone of  $(x_1, \dots, x_n)$  with non-negative entries. Let  $v := (a_1, \dots, a_n) \in \mathbb{R}^n$ , and for  $i \in \{1, \dots, n\}$  let  $C_i \subset C$  be the convex cone spanned by  $e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n$  and  $v$ . The  $C_i$ 's generate a fan in  $\mathbb{R}^n$ ;  $B_\sigma(\mathbb{A}^n)$  is the associated toric variety. Since the  $C_i$ 's define a cone decomposition of  $C$ , we have natural regular map  $\pi_\sigma: B_\sigma(\mathbb{A}^n) \rightarrow \mathbb{A}^n$ , which is an isomorphism over  $\mathbb{A}^n \setminus \{0\}$ . Let  $E_\sigma \subset B_\sigma(\mathbb{A}^n)$  be the exceptional set of  $\pi_\sigma$ ; then  $E_\sigma$  is isomorphic to the weighted projective space  $\mathbb{P}(a_1, \dots, a_n)$ . We denote by  $[x_1, \dots, x_n]$  (with  $(x_1, \dots, x_n) \neq (0, \dots, 0)$ ) a (closed) point of  $\mathbb{P}(a_1, \dots, a_n)$ ; thus  $[x_1, \dots, x_n] = [y_1, \dots, y_n]$  if and only if there exists  $t \in \mathbb{C}^*$  such that  $x_i = t^{a_i} y_i$  for  $i \in \{1, \dots, n\}$ . Actually the composition

$$\begin{array}{ccccc} B_\sigma(\mathbb{A}^n) & \xrightarrow{\pi_\sigma} & \mathbb{A}^n & \dashrightarrow & \mathbb{P}(a_1, \dots, a_n) \\ p & \mapsto & \pi_\sigma(p) = (x_1, \dots, x_n) & \mapsto & [x_1, \dots, x_n] \end{array}$$

is regular; this follows from the formulae for  $\pi_\sigma$  that follow Definition 2.1 in [And16]. Thus we have a regular map

$$(5.9) \quad B_\sigma(\mathbb{A}^n) \longrightarrow \mathbb{A}^n \times \mathbb{P}(a_1, \dots, a_n).$$

Let  $\mu_\sigma: E_\sigma \rightarrow \mathbb{P}(a_1, \dots, a_n)$  be the restriction to  $E_\sigma$  of the map in (5.9), followed by projection to the second factor. Then  $\mu_\sigma$  is an isomorphism; we will identify  $E_\sigma$  with  $\mathbb{P}(a_1, \dots, a_n)$  via  $\mu_\sigma$ . The formulae for  $\pi_\sigma$  that follow Definition 2.1 in [And16] give the following result.

**Proposition 5.1.** *Keep notation as above, and let  $\Delta \subset \mathbb{C}$  be a disc centered at 0. Let  $\alpha: \Delta \rightarrow B_\sigma(\mathbb{A}^n)$  be a holomorphic map such that  $\alpha^{-1}(E_\sigma) = \{0\}$ . There exists  $k > 0$  such that*

$$(5.10) \quad \pi_\sigma \circ \alpha(t) = (t^{ka_1} \cdot \varphi_1, \dots, t^{ka_n} \cdot \varphi_n)$$

where  $\varphi_i: \Delta \rightarrow \mathbb{C}$  is a holomorphic function, and moreover

$$(5.11) \quad \alpha(0) = [\varphi_1(0), \dots, \varphi_n(0)].$$

(In particular  $(\varphi_1(0), \dots, \varphi_n(0)) \neq (0, \dots, 0)$ .)

**Corollary 5.2.** *Let  $Z$  be a projective variety, and  $\mathbf{p}: B_\sigma(\mathbb{A}^n) \dashrightarrow Z$  be a rational map, regular away from  $E_\sigma$ . Suppose that the following holds. Given a disc  $\Delta \subset \mathbb{C}$  centered at 0, and a holomorphic map  $\alpha: \Delta \rightarrow B_\sigma(\mathbb{A}^n)$  such that  $\alpha^{-1}(E_\sigma) = \{0\}$ , the extension at 0 of the map  $\mathbf{p} \circ \alpha|_{(\Delta \setminus \{0\})}$  depends only on  $\alpha(0) = [\varphi_1(0), \dots, \varphi_n(0)]$  (notation as in (5.11)). Then  $\mathbf{p}$  is regular everywhere.*

*Proof.* Follows from **Proposition 5.1** and normality of  $B_\sigma(\mathbb{A}^n)$ .  $\square$

5.1.4. *Blow-up of the étale slice and the period map.* It will be convenient to denote by  $Z$  the affine scheme  $V(8) \oplus V(12)$ , i.e.  $Z := \text{Spec } S^\bullet(V(8)^\vee \oplus V(12)^\vee)$ . Let  $(x_1, \dots, x_9)$  be coordinates on  $V(8)$ , and  $(x_{10}, \dots, x_{22})$  be coordinates on  $V(12)$ ; then (with a slight abuse of notation)  $(x_1, \dots, x_{22})$  are coordinates on  $V(8) \oplus V(12)$ . Let  $\sigma$  be the weight defined by

$$(5.12) \quad \sigma(x_i) := \begin{cases} 4 & \text{if } i \in \{1, 9\}, \\ 6 & \text{if } i \in \{10, 22\}. \end{cases}$$

Let  $\tilde{Z} := \text{Bl}_\sigma(Z)$  be the corresponding weighted blow up, and let  $E$  be the exceptional set of  $\tilde{Z} \rightarrow Z$ ; thus  $E$  is the weighted projective space  $\mathbb{P}(4^9, 6^{13})$ . The action of  $\text{SL}_2$  on  $Z$  lifts to an action on  $\tilde{Z}$  (and on the ample line-bundle  $\mathcal{O}_{\tilde{Z}}(-E)$ ). Thus there is an associated GIT quotient  $\tilde{Z} // \text{SL}_2$ . The map  $\tilde{Z} \rightarrow Z$  induces a map

$$(5.13) \quad \tilde{\mu}: \tilde{Z} // \text{SL}_2 \longrightarrow Z // \text{SL}_2.$$

Moreover the set-theoretic inverse image  $\tilde{\mu}^{-1}([0])_{\text{red}}$  is isomorphic to  $\text{Proj } \Omega // \text{SL}_2 = \mathfrak{M}_u$ . Since the natural map  $Z // \text{SL}_2 \rightarrow \mathfrak{M}$  is dominant, it makes sense to compose it with the (rational) period map  $\mathbf{p}: \mathfrak{M} \dashrightarrow \mathcal{F}(19)^*$ . Composing with  $\tilde{\mu}$ , we get a rational map

$$(5.14) \quad \tilde{\mathbf{p}}: \tilde{Z} // \text{SL}_2 \dashrightarrow \mathcal{F}(19)^*.$$

**Theorem 5.3.** *With notation as above, the map  $\tilde{\mathbf{p}}$  is regular in a neighborhood of  $\tilde{\mu}^{-1}([0])_{\text{red}} = \mathfrak{M}_u$ , and its restriction to  $\tilde{\mu}^{-1}([0])_{\text{red}}$  is equal to the period map  $\mathbf{p}_u$  in (5.8).*

*Proof.* This follows from the results of Shah in [Sha81]. More precisely, let  $(F, G) \in V(8) \oplus V(12)$  be non-zero and such that  $[(F, G)] \in \text{Proj } \Omega$  is  $\text{SL}_2$ -semistable. Let  $\Delta \subset \mathbb{C}$  be a disc centered at 0, and

$$(5.15) \quad \begin{array}{ccc} \Delta & \xrightarrow{\varphi} & V(8) \oplus V(12) \\ t & \mapsto & (t^{4m} F(t), t^{6m} G(t)) \end{array}$$

where  $m > 0$ ,  $F(t)$ ,  $G(t)$  are holomorphic, and  $F(0) = F$ ,  $G(0) = G$ . (This is the family on the second-to-last displayed equation of p. 293, with the difference that our  $(0, 0) \in Z$  corresponds to Shah's  $F_0$ .) We assume also that for  $t \neq 0$ , the point  $[\varphi(t)]$  is *not* in the indeterminacy locus of the period map  $Z // \text{SL}_2 \dashrightarrow \mathcal{F}(19)^*$ . Let  $\mathbf{p}_\varphi: \Delta \rightarrow \mathcal{F}(19)^*$  be the holomorphic extension of the composition  $(\Delta \setminus \{0\}) \rightarrow Z // \text{SL}_2 \dashrightarrow \mathcal{F}(19)^*$ . Then by Theorem 3.17 of [Sha81], the value  $\mathbf{p}_\varphi(0)$

is equal to the period point  $\mathfrak{p}_u([(F, G)])$ . Thus  $\mathfrak{p}$  is regular in a neighborhood of  $\tilde{\mu}^{-1}([0])_{\text{red}} = \mathfrak{M}_u$  by **Corollary 5.2**, and the restriction of the period map to  $\tilde{\mu}^{-1}([0])_{\text{red}}$  is equal to the period map  $\mathfrak{p}_u$  in (5.8).  $\square$

**5.1.5. Blow-up of  $\mathfrak{M}$  at  $v$ .** A weighted blow up  $\text{Bl}_\sigma(\mathbb{A}^n) \rightarrow \mathbb{A}^n$  is equal to the blow up of a suitable scheme supported at 0, see Remark 2.5 of [And16]. It follows that also the map in (5.13) is the blow up of an ideal  $\mathcal{J}$  supported on  $[0]$ . Since the map in (5.4) is an isomorphism of analytic germs, the ideal sheaf  $\mathcal{J}$  defines an ideal sheaf in  $\mathcal{O}_{\mathfrak{M}}$ , cosupported at  $v$ , that we will denote by  $\mathcal{I}$ . Let  $\mathfrak{M}_v := \text{Bl}_{\mathcal{I}} \mathfrak{M}$ , and let  $E_v \subset \mathfrak{M}_v$  be the (reduced) exceptional divisor of  $\text{Bl}_{\mathcal{I}} \mathfrak{M} \rightarrow \mathfrak{M}$ . Thus  $E_v \cong \mathfrak{M}_u$ , and  $E_v$  is  $\mathbb{Q}$ -Cartier. Let  $\phi_v: \mathfrak{M}_v \rightarrow \mathfrak{M}$  be the natural map. By **Theorem 5.3**, the period map  $\mathfrak{M}_v \dashrightarrow \mathcal{F}(19)^*$  is regular in a neighborhood of  $E_v$ . Moreover, letting  $\mathcal{L}$  be the ample  $\mathbb{Q}$ -line bundle on  $\mathfrak{M}$  descended from the ample generator of  $|\mathcal{O}_{\mathbb{P}^3}(4)|$ , the line-bundle  $\phi_v^* \mathcal{L}(-\epsilon E(v))$  is ample for  $\epsilon$  positive and sufficiently small.

## 5.2. Blow up of the point $\omega$ .

**5.2.1. The GIT moduli space for hyperelliptic quartic  $K3$  surfaces.** The hyperelliptic GIT moduli space that we will consider is

$$(5.16) \quad \mathfrak{M}_h := |\mathcal{O}_{\mathbb{P}^1}(4) \boxtimes \mathcal{O}_{\mathbb{P}^1}(4)| // \text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1).$$

Given  $D \in |\mathcal{O}_{\mathbb{P}^1}(4) \boxtimes \mathcal{O}_{\mathbb{P}^1}(4)|$ , we let  $\pi: X_D \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  be the double cover ramified over  $D$ , and  $L_D := \pi^* \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$ . If  $D$  has ADE singularities, then  $(X_D, L_D)$  is a hyperelliptic quartic  $K3$ . We recall that if  $(X, L)$  is a hyperelliptic quartic  $K3$  surface, the map  $\varphi_L$  associated to the complete linear system  $|L| \cong \mathbb{P}^3$  is regular, and it is the double cover of an irreducible quadric  $Q$ , branched over a divisor  $B \in |\mathcal{O}_Q(4)|$  with ADE singularities. Vice versa, the double cover of an irreducible quadric surface,  $Q \subset \mathbb{P}^3$ , branched over a divisor  $B \in |\mathcal{O}_Q(4)|$  with ADE singularities is a hyperelliptic quartic  $K3$  surface. The period space for  $\mathfrak{M}_h$  is  $\mathcal{F}_h$ ; we let

$$(5.17) \quad \mathfrak{p}_h: \mathfrak{M}_h \dashrightarrow \mathcal{F}_h^*$$

be the extension of the period map to the Baily-Borel compactification.

**Theorem 5.4.** (1) *A divisor in  $|\mathcal{O}_{\mathbb{P}^1}(4) \boxtimes \mathcal{O}_{\mathbb{P}^1}(4)|$  with ADE singularities is  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  stable, hence there exists an open dense subset  $\mathfrak{M}_h^I \subset \mathfrak{M}_h$  parametrizing isomorphism classes of hyperelliptic quartic  $K3$  surfaces such that  $\varphi_L(X)$  is a smooth quadric.*  
(2) *The period map  $\mathfrak{p}_h$  defines an isomorphism between  $\mathfrak{M}_h^I$  and the complement of the “hyperelliptic” divisor  $H_h(\mathcal{F}_h)$  in  $\mathcal{F}_h$  (the divisor  $H_h(18) \subset \mathcal{F}(18)$  in the notation of [LO16]).*

*Proof.* Item (1) is a result of Shah, in fact it is contained in Theorem 4.8 of [Sha81]. Item (2) follows from the discussion above. In fact let  $y \in \mathfrak{F}_h$ . Then there exists a hyperelliptic quartic  $K3$  surface  $(X, L)$  (unique up to isomorphism) whose period point is  $y$ , and the quadric  $Q := \varphi_L(X)$  is smooth if and only if  $y \notin H_h(\mathcal{F}_h)$ .  $\square$

**5.2.2. The germ of  $\mathfrak{M}$  at  $\omega$  in the analytic topology.** Let  $q \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$  be a non degenerate quadratic form, and let  $Q \subset \mathbb{P}^3$  be the smooth quadric with equation  $q = 0$ . Let  $O(q)$  be the associated orthogonal group; then  $\text{PO}(q) = \text{Aut} Q$  is the stabilizer of  $[q^2] \in |\mathcal{O}_{\mathbb{P}^3}(4)|$ . We have a decomposition of  $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$  into  $O(q)$ -modules

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) = q \cdot H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \oplus H^0(Q, \mathcal{O}_Q(2)).$$

Note that the first submodule is reducible (it contains a trivial summand, spanned by  $q^2$ ), while the second one is irreducible. We identify  $Q$  with  $\mathbb{P}^1 \times \mathbb{P}^1$ ,  $\text{PO}(q)$  with  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ , and  $H^0(Q, \mathcal{O}_Q(2))$  with  $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4) \boxtimes \mathcal{O}_{\mathbb{P}^1}(4))$ . The projectivization of  $q \cdot H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$  is equal to the

projective (embedded) tangent space at  $[q^2]$  of the orbit  $\mathrm{PGL}(4)[q^2]$ . Thus, by Luna's étale slice Theorem, we have natural étale map

$$H^0(Q, \mathcal{O}_Q(2)) // O(q) \longrightarrow \mathfrak{M},$$

mapping  $[0]$  to  $\omega$ . In particular we have an isomorphism of analytic germs

$$(5.18) \quad (H^0(Q, \mathcal{O}_Q(2)) // O(q), [0]) \xrightarrow{\sim} (\mathfrak{M}, \omega).$$

**5.2.3. Partial extension of the period map on the blow up of  $\omega$ .** The map  $\phi_v: \mathfrak{M}_v \rightarrow \mathfrak{M}$  is an isomorphism over  $\mathfrak{M} \setminus \{v\}$ ; abusing notation, we denote by the same symbol  $\omega$  the unique point in  $\mathfrak{M}_v$  lying over  $\omega \in \mathfrak{M}$ . Let  $\phi_\omega: \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}_v$  be the blow-up of the reduced point  $\omega$ , and let  $E_\omega \subset \widetilde{\mathfrak{M}}$  be the exceptional divisor. We let  $\phi := \phi_v \circ \phi_\omega$ , and  $\tilde{\mathfrak{p}} = \mathfrak{p} \circ \phi$ . Thus we have

$$\begin{array}{ccc} & \widetilde{\mathfrak{M}} & \\ \phi \swarrow & & \searrow \tilde{\mathfrak{p}} \\ \mathfrak{M} & & \mathcal{F}^* \end{array}$$

**Proposition 5.5.** *Keeping notation as above,  $E_\omega$  is naturally identified with the hyperelliptic GIT moduli space  $\mathfrak{M}_h$ , and the restriction of  $\tilde{\mathfrak{p}}$  to  $E_\omega$  is equal to the period map  $\mathfrak{p}_h$  of (5.17).*

*Proof.* Let  $\psi_\omega: \mathfrak{M}_\omega \rightarrow \mathfrak{M}$  be the blow-up of the reduced point  $\omega$ , and let  $\mathfrak{p}_\omega: \mathfrak{M}_\omega \rightarrow \mathcal{F}^*$  be the composition  $\mathfrak{p} \circ \psi_\omega$ . Since  $\omega$  and  $v$  are disjoint subschemes of  $\mathfrak{M}$ , the exceptional divisor of  $\psi_\omega$  is identified with  $E_\omega$ , and it suffices to prove that Items (1) and (2) hold with  $\widetilde{\mathfrak{M}}$  and  $\tilde{\mathfrak{p}}$  replaced by  $\mathfrak{M}_\omega$  and  $\mathfrak{p}_\omega$  respectively. Let  $\mathbf{D} \subset |\mathcal{O}_{\mathbb{P}^3}(4)|$  be the closed subset of double quadrics, i.e. the closure of the orbit  $\mathrm{PGL}(4)(2Q)$ , where  $Q \subset \mathbb{P}^3$  is a smooth quadric. Let  $\pi: P \rightarrow |\mathcal{O}_{\mathbb{P}^3}(4)|$  be the blow up of (the reduced)  $\mathbf{D}$ , and let  $E_{\mathbf{D}} \subset P$  be the exceptional divisor of  $\pi$ . Then  $\mathrm{PGL}(4)$  acts on  $P$  (because  $\mathbf{D}$  is  $\mathrm{PGL}(4)$ -invariant), and the action lifts to an action on the line bundle  $\mathcal{O}_P(E_{\mathbf{D}})$ . Let  $\mathcal{L}$  be the hyperplane line bundle on  $|\mathcal{O}_{\mathbb{P}^3}(4)|$ , and let  $t \in \mathbb{Q}_+$  be such that  $f^*\mathcal{L}(-tE_{\mathbf{D}})$  is an ample  $\mathbb{Q}$ -line bundle on  $P$ . Then  $\mathrm{PGL}(4)$  acts on the ring of global sections  $R(P, \pi^*\mathcal{L}(-tE_{\mathbf{D}}))$ , and hence we may consider the GIT moduli space

$$\widehat{\mathfrak{M}}(t) := \mathrm{Proj} \left( R(P, \pi^*\mathcal{L}(-tE_{\mathbf{D}}))^{\mathrm{PGL}(4)} \right).$$

By Kirwan [Kir85], there exists  $t_0 > 0$  such that the blow down map  $\pi: P \rightarrow |\mathcal{O}_{\mathbb{P}^3}(4)|$  induces a regular map  $\widehat{\psi}(t): \widehat{\mathfrak{M}}(t) \rightarrow \mathfrak{M}$  for all  $0 < t < t_0$ , and moreover  $\widehat{\mathfrak{M}}(t)$  and  $\widehat{\psi}(t)$  are identified with  $\mathfrak{M}_\omega$  and  $\psi_\omega$  respectively. But now the identification of  $E_\omega$  with the hyperelliptic GIT moduli space  $\mathfrak{M}_h$  follows at once from the isomorphism of germs in (5.18). The assertion on the period map follows from the description of the germ  $(\mathfrak{M}, \omega)$  and a standard semistable replacement argument.  $\square$

**5.3. Identification of  $\mathcal{F}(1 - \epsilon)$  and  $\widetilde{\mathfrak{M}}$ .** Let  $\mathcal{L}$  be the  $\mathbb{Q}$  line bundle on  $\mathfrak{M}$  induced by the hyperplane line bundle on  $|\mathcal{O}_{\mathbb{P}^3}(4)|$ , and let  $\widetilde{\mathcal{L}} := \phi^*\mathcal{L}$ . Let  $E := E_v + E_\omega$ . Then

$$(5.19) \quad (\tilde{\mathfrak{p}}^{-1})^*(E)|_{\mathcal{F}} = H_h + H_u = 2\Delta.$$

In fact we have the set-theoretic equalities  $\tilde{\mathfrak{p}}(E_v) \cap \mathcal{F} = H_u$ , and  $\tilde{\mathfrak{p}}(E_\omega \setminus \mathrm{Ind}(\tilde{\mathfrak{p}})) \cap \mathcal{F} = H_h \setminus H_h^{(2)}$ , thus in order to finish the proof of (5.19) one only needs to compute multiplicities; they are equal to 1 because  $\tilde{\mathfrak{p}}^{-1}$  has degree 1. By (5.19) and Equation (4.1.2) of [LO16], we get

$$(\tilde{\mathfrak{p}}^{-1})^*(\widetilde{\mathcal{L}}(-\epsilon E))|_{\mathcal{F}} \cong \mathcal{O}_{\mathcal{F}}(\lambda + (1 - 2\epsilon)\Delta).$$

Thus  $\tilde{\mathfrak{p}}^{-1}$  induces a homomorphism

$$(5.20) \quad R(\mathfrak{M}, \widetilde{\mathcal{L}}(-\epsilon E)) \longrightarrow R(\mathcal{F}, \lambda + (1 - 2\epsilon)\Delta).$$

**Proposition 5.6.** *The homomorphism in (5.20) is an isomorphism of rings.*

*Proof.* This is because  $\tilde{\mathfrak{p}}^{-1}$  is an isomorphism between  $\mathcal{F} \setminus H_h^{(2)}$ , which has complement of codimension 2 in  $\mathcal{F}$ , and an open subset of  $\widetilde{\mathfrak{M}}$  which again has complement of dimension 2 in  $\widetilde{\mathfrak{M}}$ .  $\square$

**Corollary 5.7.** *The restriction of  $\tilde{\mathfrak{p}}^{-1}$  to  $\mathcal{F}$  defines an isomorphism*

$$\mathrm{Proj}(\mathcal{F}, \lambda + (1 - \epsilon)\Delta) \cong \widetilde{\mathfrak{M}}$$

for small enough  $0 < \epsilon$ .

*Proof.* If  $0 < \epsilon$  is small enough, then  $\widetilde{\mathcal{L}}(-\epsilon E)$  is ample on  $\widetilde{\mathfrak{M}}$ , and hence  $\mathrm{Proj}R(\mathfrak{M}, \widetilde{\mathcal{L}}(-\epsilon E)) \cong \widetilde{\mathfrak{M}}$ . Thus the corollary follows from **Proposition 5.6**.  $\square$

**5.4. The first flip of the GIT quotient ( $\beta = 1/2$ ).** We recall that the curve  $W_1 \subset \mathfrak{M}$  contains the point  $\omega$  and does not contain  $v$ . We let  $\widetilde{W}_1 \subset \widetilde{\mathfrak{M}}$  be the strict transform of  $W_1$ . We will perform a surgery of  $\mathfrak{M}$  along  $\widetilde{W}_1$  in order to obtain our candidate for  $\mathcal{F}(1/3, 1/2)$ , notation as in (1.10). More precisely, we will start by constructing a birational map  $\widehat{\mathfrak{M}} \rightarrow \widetilde{\mathfrak{M}}$ , which is an isomorphism away from  $\widetilde{W}_1$ , and over  $\widetilde{W}_1$  is a weighted blow along normal slices to  $\widetilde{W}_1$ . Let  $E_1$  be the exceptional divisor of  $\widehat{\mathfrak{M}} \rightarrow \widetilde{\mathfrak{M}}$ ; then  $E_1 \cong \widetilde{W}_1 \times \mathfrak{M}_c$ , where  $\mathfrak{M}_c$  is a GIT compactification of the moduli space of degree-4 polarized  $K3$  surfaces which are double covers of a quadric cone with branch divisor not containing the vertex of the cone. Let  $\widehat{\mathfrak{p}}: \widehat{\mathfrak{M}} \dashrightarrow \mathcal{F}$  be the period map and  $\widehat{\mathfrak{M}}_{\mathrm{reg}} \subset \widehat{\mathfrak{M}}$  be the subset of regular points of  $\widehat{\mathfrak{p}}$ ; we will show that, if  $p \in \widetilde{W}_1$ , then the intersection  $\widehat{\mathfrak{M}}_{\mathrm{reg}} \cap \{p\} \times \mathfrak{M}_c$  (here  $\{p\} \times \mathfrak{M}_c \subset E_1$ ) coincides with the set of regular points of the period map  $\mathfrak{M}_c \dashrightarrow \mathcal{F}$ , and that the restriction of  $\widehat{\mathfrak{p}}$  is equal to the period map  $\mathfrak{M}_c \dashrightarrow \mathcal{F}$ . It follows that  $\widehat{\mathfrak{p}}$  is constant on the slices  $\{p\} \times \mathfrak{M}_c \subset E_1$ , and the image of the restriction of  $\widehat{\mathfrak{p}}$  to the set of regular points of  $E_1$  is the complement of  $\Delta^{(3)} = \mathrm{Im}(f_{16,19})$  in the codimension-2 locus  $\Delta^{(2)} = \mathrm{Im}(f_{17,19})$  (notation as in [LO16]). Now,  $\widehat{\mathfrak{M}}$  can be contracted along  $E_1 \rightarrow \mathfrak{M}_c$ , let  $\mathfrak{M}_{1/2}$  be the contraction; the results mentioned above strongly suggest that  $\mathfrak{M}_{1/2}$  is isomorphic to  $\mathcal{F}(1/3, 1/2)$ .

**5.4.1. The action on quartics of the automorphism group of polystable surfaces in  $W_1$ .** Let  $q := x_0^2 + x_1^2 + x_2^2$ , and let

$$(5.21) \quad f_{a,b} := (q + ax_3^2)(q + bx_3^2),$$

where  $(a, b) \neq (0, 0)$ . Then  $V(f_{a,b})$  is a polystable quartic, and its equivalence class belongs to  $W_1$ . Conversely, if  $V(f)$  is a polystable quartic whose equivalence class belongs to  $W_1$ , then up to projectivities and rescaling,  $f = f_{a,b}$  for some  $(a, b) \neq (0, 0)$ . The points in  $\mathfrak{M}$  representing  $V(f_{a,b})$  and  $V(f_{c,d})$  are equal if and only if  $[a, b] = [c, d]$ , or  $[a, b] = [d, c]$ . Lastly,  $V(f_{a,b})$  represents  $\omega$  if and only if  $a = b$ .

Suppose that  $a \neq b$ . Then every element of  $\mathrm{Aut}V(f_{a,b})$  fixes  $V(x_3)$  and the point  $[0, 0, 0, 1]$ . It follows that  $\mathrm{Aut}V(f_{a,b})$  is equal to the image of the natural map  $\mathrm{O}(q) \rightarrow \mathrm{PGL}(4)$ . In particular  $\mathrm{SO}(q)$  is an index 2 subgroup of  $\mathrm{Aut}V(f_{a,b})$ , and hence the double cover of  $\mathrm{SO}(q)$ , i.e.  $\mathrm{SL}_2$ , acts on  $V(f_{a,b})$ . The decomposition into irreducible representations of the action of  $\mathrm{SL}_2$  on  $\mathbb{C}[x_0, \dots, x_3]_4$  is as follows:

$$(5.22) \quad \begin{array}{cccccc} \mathbb{C}[x_0, \dots, x_2]_4 & \oplus & \mathbb{C}[x_0, \dots, x_2]_3 \cdot x_3 & \oplus & \mathbb{C}[x_0, \dots, x_2]_2 \cdot x_3^2 & \oplus & \mathbb{C}[x_0, \dots, x_2]_1 \cdot x_3^3 & \oplus & \mathbb{C} \cdot x_3^4 \\ V(8) \oplus V(4) \oplus V(0) & & V(6) \oplus V(2) & & V(4) \oplus V(0) & & V(2) & & V(0) \end{array}$$

Now let us determine the sub-representation  $U_{a,b}$  containing  $[f_{a,b}]$  and such that  $\mathrm{Hom}([f_{a,b}], U/[f_{a,b}])$  is the tangent space at  $V(f_{a,b})$  to the orbit  $\mathrm{PGL}(4)V(f_{a,b})$  (we only assume that  $(a, b) \neq (0, 0)$ ). Let  $\ell_i \in \mathbb{C}[x_0, \dots, x_3]_1$  for  $i \in \{0, \dots, 3\}$ ; we pick up the term multiplying  $t$  in the expansion of  $f_{a,b}(x_0 + t\ell_0, \dots, x_3 + t\ell_3)$  as element of  $\mathbb{C}[x_0, \dots, x_3]_4[t]$ . Letting  $\ell_i = \mu_i x_i$ , we get

$$(5.23) \quad 4q \left( \sum_{i=0}^2 \mu_i x_i \right) x_3 + 2(a+b)\mu_3 q x_3^2 + 2(a+b) \left( \sum_{i=0}^2 \mu_i x_i \right) x_3^3 + 4ab\mu_3 x_3^4.$$



Letting  $\ell_i \in \mathbb{C}[x_0, x_1, x_2]_1$ , we get

$$(5.24) \quad 4q \left( \sum_{i=0}^2 \ell_i x_i \right) + 2(a+b)q\ell_3 x_3 + 2(a+b) \left( \sum_{i=0}^2 \ell_i x_i \right) x_3^2 + 4ab\ell_3 x_3^3.$$

It follows that

$$(5.25) \quad U_{a,b} \cong \begin{cases} V(4) \oplus V(2)^2 \oplus V(0)^2 & \text{if } a \neq b, \\ V(4) \oplus V(2) \oplus V(0)^2 & \text{if } a = b. \end{cases}$$

The difference between the two cases is due to the different behaviour of the  $V(2)$ -representations appearing in (5.23), (5.24) and contained in the direct sum  $\mathbb{C}[x_0, \dots, x_2]_3 \cdot x_3 \oplus \mathbb{C}[x_0, \dots, x_2]_1 \cdot x_3^3$ . If  $a \neq b$ , the representations in (5.23) and (5.24) are distinct, if  $a = b$  they are equal.

**5.4.2. The germ of  $\widetilde{\mathfrak{M}}$  at points of  $\widetilde{W}_1 \setminus E_\omega$ .** The map  $\phi: \widetilde{\mathfrak{M}} \rightarrow \mathfrak{M}$  is an isomorphism away from  $\{\omega, v\}$ . Since  $W_1$  does not contain  $v$ , the germ of  $\mathfrak{M}$  at a point  $\tilde{x} \in (\widetilde{W}_1 \setminus E_\omega)$  is identified by  $\phi$  with the germ of  $\mathfrak{M}$  at  $x := \phi(\tilde{x})$ . Thus we will examine the germ of  $\mathfrak{M}$  at a point  $x \in (W_1 \setminus \{\omega\})$ . There exists  $(a, b) \in \mathbb{C}^2$ , with  $a \neq b$ , such that a polystable quartic representing  $x$  is  $V(f_{a,b})$ , where  $f_{a,b}$  is as in (5.21). Keeping notation as in **Subsubsection 5.4.1**,  $\text{SL}_2$  acts on  $V(f_{a,b})$ . Let  $N_{a,b} \subset \mathbb{C}[x_0, \dots, x_3]_4$  be the sub  $\text{SL}_2$ -representation

$$(5.26) \quad N_{a,b} := V(8) \oplus V(6) \oplus R \cdot x_3^2 \oplus \langle 2qx_3^2 + (a+b)x_3^4 \rangle,$$

where  $R \subset \mathbb{C}[x_0, x_1, x_2]_2$  is the summand isomorphic to  $V(4)$ , and let

$$(5.27) \quad \mathbf{N}_{a,b} := \{V(f_{a,b} + g) \mid g \in N_{a,b}\}.$$

**Proposition 5.8.** *Keeping notation as above,  $\mathbf{N}_{a,b}$  is an  $\text{Aut}V(f_{a,b})$ -invariant normal slice to the orbit  $\text{PGL}(4)V(f_{a,b})$ .*

*Proof.* Let  $U_{a,b} \subset \mathbb{C}[x_0, \dots, x_3]_4$  be as in **Subsubsection 5.4.1**; thus  $\mathbb{P}(U_{a,b})$  is the projective tangent space at  $V(f_{ab})$  to the orbit  $\text{PGL}(4)V(f_{a,b})$ . Then  $U_{a,b}$  is the sum of the two  $\text{SL}_2$ -representations in (5.23) and (5.24) (and, as representation, it is given by the first case in (5.25)), and it follows that the  $\text{SL}_2$ -invariant affine space in (5.27) is transversal to  $\mathbb{P}(U_{a,b})$  at  $V(f_{a,b})$ . Lastly,  $\mathbf{N}_{a,b}$  is  $\text{Aut}V(f_{a,b})$ -invariant because  $\text{Aut}V(f_{a,b})$  is generated by the image of  $\text{SL}_2$  and the reflection in the plane  $x_3 = 0$ .  $\square$

The natural map

$$(5.28) \quad \psi: \mathbf{N}_{a,b} // \text{Aut}V(f_{a,b}) \longrightarrow \mathfrak{M}$$

is étale at  $V(f_{a,b})$  by Luna's étale slice Theorem. For later use, we make the following observation.

**Claim 5.9.** *Keep notation and assumptions as above, in particular  $a \neq b$ . Let  $\eta: \mathbf{N}_{a,b} \rightarrow \mathfrak{M}$  be the composition of the quotient map  $\mathbf{N}_{a,b} \rightarrow \mathbf{N}_{a,b} // \text{Aut}V(f_{a,b})$  and the map  $\psi$  in (5.28). Then*

$$(5.29) \quad \eta(\{V(f_{a,b} + t(2qx_3^2 + (a+b)x_3^4)) \mid t \in \mathbb{C}\}) \subset W_1.$$

*Moreover, let  $\mathcal{U} \subset \mathbf{N}_{a,b}$  be an  $\text{Aut}V(f_{a,b})$ -invariant open (in the classical topology) neighborhood of  $f_{a,b}$  such that the restriction of  $\psi$  to  $\mathcal{U} // \text{Aut}V(f_{a,b})$  is an isomorphism onto  $\psi(\mathcal{U} // \text{Aut}V(f_{a,b}))$ ; then  $x \in \mathcal{U}$  is mapped to  $W_1$  by  $\eta$  and has closed  $\text{SL}_2$ -orbit if and only if  $x = V(f_{a,b} + t(2qx_3^2 + (a+b)x_3^4))$  for some  $t \in \mathbb{C}$ .*

*Proof.* The first statement follows from a direct computation. In fact, an easy argument shows that there exist holomorphic functions  $\varphi, \psi$  of the complex variable  $t$  vanishing at  $t = 0$ , such that

$$(q + (a + \varphi(t))x_3^2) \cdot (q + (b + \psi(t))x_3^2) = f_{a,b} + t(2qx_3^2 + (a+b)x_3^4).$$

The second statement holds because  $W_1$  is an irreducible curve, and so is the left-hand side of (5.29).  $\square$

5.4.3. *The germ of  $\widetilde{\mathfrak{M}}$  at the unique point in  $\widetilde{W}_1 \cap E_\omega$ .* Let  $\mathfrak{M}_\omega \rightarrow \mathfrak{M}$  be the blow-up of (the reduced)  $\omega$ . We may work on  $\mathfrak{M}_\omega$ , since  $W_1$  does not contain  $v$ . Let  $P \rightarrow |\mathcal{O}_{\mathbb{P}^3}(4)|$  be the blow-up with center the closed subset  $\mathbf{D}$  parametrizing double quadrics, and let  $E_{\mathbf{D}}$  be the exceptional divisor. By [Kir85] the blow-up  $\mathfrak{M}_\omega$  is identified with the quotient of  $P$  by the natural action of  $\mathrm{PGL}(4)$  (with a polarization close to the pull-back of the hyperplane line-bundle on  $|\mathcal{O}_{\mathbb{P}^3}(4)|$ ) - see the proof of **Proposition 5.5**. We will describe an  $\mathrm{SL}_2$ -invariant normal slice in  $P$  to the  $\mathrm{PGL}(4)$ -orbit of a point representing the unique point in  $\widetilde{W}_1 \cap E_\omega$ . First, recall that we have an identification  $E_\omega = \mathfrak{M}_h$ , where  $\mathfrak{M}_h$  is the GIT hyperelliptic moduli space in (5.16), see **Proposition 5.5**. The unique point in  $\widetilde{W}_1 \cap E_\omega$  is represented by a point in  $E_{\mathbf{D}}$  mapping to a smooth quadric  $Q \subset \mathbb{P}^3$ , and corresponding to  $\ell^4 \in \mathbb{P}(H^0(\mathcal{O}_Q(4)))$  (recall that the fiber of the exceptional divisor over  $Q$  is identified with  $\mathbb{P}(H^0(\mathcal{O}_Q(4)))$ , where  $0 \neq \ell \in H^0(\mathcal{O}_Q(1))$  is a section with *smooth* zero-locus (a smooth conic); moreover the points we have described have closed orbit in the locus of  $\mathrm{PGL}(4)$ -semistable points.

**Notation/Convention 5.10.** We represent the unique point in  $\widetilde{W}_1 \cap E_\omega$  by the point with closed orbit  $(V(q + ax_3^2), x_3^4) \in E_{\mathbf{D}}$  (notation as above), where  $q$  is as in **Subsubsection 5.4.1** and  $a \neq 0$ . In order to simplify notation, we let  $Q_a := V(q + ax_3^2)$ , and  $p := (Q_a, x_3^4) \in E_{\mathbf{D}}$ .

Now let  $S \subset \mathbb{C}[x_0, \dots, x_3]_4$  be the sub  $\mathrm{SL}_2$ -representation

$$(5.30) \quad S := V(8) \oplus V(6) \oplus R \cdot x_3^2 \oplus \langle x_3^4 \rangle,$$

where  $R \subset \mathbb{C}[x_0, x_1, x_2]_2$  is the summand isomorphic to  $V(4)$ . (Notice the similarity with (5.26).) Let

$$(5.31) \quad \mathbf{S}_a := \{V(f_{a,a} + g) \mid g \in S\}$$

**Claim 5.11.** *Keeping notation as above, the double quadric  $V(f_{a,a})$  is an isolated and reduced point of the scheme-theoretic intersection between the affine space  $\mathbf{S}_a$  and the closed  $\mathbf{D} \subset |\mathcal{O}_{\mathbb{P}^3}(4)|$  parametrizing double quadrics.*

*Proof.* Of course  $V(f_{a,a}) \in \mathbf{D}$ , because  $f_{a,a} = (q + ax_3^2)^2$ . Let  $T_{V(f_{a,a})}\mathbf{S}_a$  and  $T_{V(f_{a,a})}\mathbf{D}$  be the tangent spaces to  $\mathbf{S}_a$  and  $\mathbf{D}$  at  $V(f_{a,a})$  respectively; we must show that their intersection (as subspaces of  $T_{V(f_{a,a})}|\mathcal{O}_{\mathbb{P}^3}(4)|$ ) is trivial. We have

$$T_{V(f_{a,a})}\mathbf{S}_a = \mathrm{Hom}(\langle f_{a,a} \rangle, \langle S, f_{a,a} \rangle / \langle f_{a,a} \rangle), \quad T_{V(f_{a,a})}\mathbf{D} = \mathrm{Hom}(\langle f_{a,a} \rangle, U_{a,a} / \langle f_{a,a} \rangle),$$

where  $U_{a,a}$  is as in **Subsubsection 5.4.1**. As is easily checked,

$$(5.32) \quad S \cap U_{a,a} = \{0\}.$$

Thus  $\langle S, f_{a,a} \rangle \cap U_{a,a} = \langle f_{a,a} \rangle$ , and the claim follows.  $\square$

By **Claim 5.11** the scheme-theoretic intersection  $\mathbf{D} \cap \mathbf{S}_a$  is the disjoint union of the reduced singleton  $\{V(f_{a,a})\}$  and a subscheme  $Y_a$ . Let  $\mathbf{U}_a := \mathbf{S}_a \setminus Y_a$ ; then  $\mathbf{U}_a$  is an open neighborhood of  $V(f_{a,a})$  in  $\mathbf{S}_a$ , and it is invariant under the action of  $\mathrm{Aut}V(f_{a,a})$ . Let  $\widetilde{\mathbf{U}}_a \subset P$  be the strict transform of  $\mathbf{U}_a$  (recall that  $P \rightarrow |\mathcal{O}_{\mathbb{P}^3}(4)|$  is the blow-up with center  $\mathbf{D}$ ), and let  $\varphi: \widetilde{\mathbf{U}}_a \rightarrow \mathbf{U}_a$  be the restriction of the contraction  $P \rightarrow |\mathcal{O}_{\mathbb{P}^3}(4)|$ . By **Claim 5.11**  $\varphi$  is the blow-up of the (reduced) point  $V(f_{a,a})$ .

*Remark 5.12.* Since  $f_{a,a}, x_3^4 \in \mathbf{S}_a$ , the point  $p = (Q_a, x_3^4) \in E_{\mathbf{D}}$  (see **Notation/Convention 5.10**) belongs to  $\widetilde{\mathbf{U}}_a$ . Moreover the stabilizer (in  $\mathrm{PGL}(4)$ ) of  $p$  is equal to  $O(q)$  i.e. to  $\mathrm{Aut}V(f_{a,b})$  for  $a \neq b$  (see **Subsubsection 5.4.1**), and it preserves  $\widetilde{\mathbf{U}}_a$ .

**Proposition 5.13.** *Keeping notation as above,  $\widetilde{\mathbf{U}}_a$  is a  $\mathrm{Stab}(p)$ -invariant normal slice to the orbit  $\mathrm{PGL}(4)p$  in  $P$ .*

*Proof.* Let  $Y := \mathrm{PGL}(4)p$ . We must prove that the tangent space to  $\tilde{\mathbf{U}}_a$  at  $p$  is transversal to the tangent space to  $Y$  at  $p$ . First notice that  $\dim Y = 12$  and  $\dim \tilde{\mathbf{U}}_a = 22$ , hence  $\dim Y + \dim \tilde{\mathbf{U}}_a = \dim P$ . Thus it suffices to prove that

$$(5.33) \quad T_p Y \cap T_p \tilde{\mathbf{U}}_a = \{0\}.$$

Let  $\pi: P \rightarrow |\mathcal{O}_{\mathbb{P}^3}(4)|$  be the blow up of  $\mathbf{D}$ . By **Claim 5.11**,

$$d\pi(p)(T_p \tilde{\mathbf{U}}_a) = \mathrm{Hom}(\langle f_{a,a} \rangle, \langle f_{a,a}, x_3^4 \rangle / \langle f_{a,a} \rangle).$$

On the other hand,  $d\pi(p)(T_p Y) = T_{\pi(p)} \mathbf{D}$ , and hence  $d\pi(p)(T_p \tilde{\mathbf{U}}_a) \cap d\pi(p)(T_p Y) = \{0\}$ . It follows that the intersection on the left hand side of (5.33) is contained in the kernel of the restriction of  $d\pi(p)$  to  $T_p \tilde{\mathbf{U}}_a$ , i.e.  $T_p(\tilde{\mathbf{U}}_a \cap E_{\pi(p)})$ , where  $E_{\pi(p)}$  is the fiber of  $E_{\mathbf{D}} \rightarrow \mathbf{D}$  over  $\pi(p) = V(f_{a,a})$ . Hence it suffices to prove that

$$(5.34) \quad T_p Y \cap T_p(\tilde{\mathbf{U}}_a \cap E_{\pi(p)}) = \{0\}.$$

The fiber  $E_{\pi(p)}$  is naturally identified with  $\mathbb{P}H^0(\mathcal{O}_{Q_a}(4))$ . With this identification, we have

$$\begin{aligned} T_p Y \cap T_p E_{\pi(p)} &= \mathrm{Hom}(\langle x_3^4 \rangle, \mathbb{C}[x_0, \dots, x_3]_1 \cdot x_3^3 / \langle x_3^4 \rangle), \\ T_p(\tilde{\mathbf{U}}_a \cap E_{\pi(p)}) &= \mathrm{Hom}(\langle x_3^4 \rangle, S / \langle x_3^4 \rangle). \end{aligned}$$

Here we are abusing notation:  $\mathbb{C}[x_0, \dots, x_3]_1 \cdot x_3^3$  and  $S$  stand for their images in  $H^0(\mathcal{O}_{Q_a}(4))$ . Since the kernel of the restriction map  $H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \rightarrow H^0(\mathcal{O}_{Q_a}(4))$  is equal to  $U_{a,a}$ , Equation (5.34) follows from the equalities

$$\begin{aligned} \langle \mathbb{C}[x_0, \dots, x_3]_1 \cdot x_3^3, S \rangle \cap U_{a,a} &= \{0\}, \\ (\mathbb{C}[x_0, \dots, x_3]_1 \cdot x_3^3) \cap S &= \langle x_3^4 \rangle \end{aligned}$$

□

The natural map

$$(5.35) \quad \psi: \tilde{\mathbf{U}}_a // \mathrm{Stab}(p) \longrightarrow \mathfrak{M}$$

is étale at  $p$  by Luna's étale slice Theorem. The result below is the analogue of **Claim 5.9**.

**Claim 5.14.** *Keep notation and assumptions as above. Let  $\zeta: \tilde{\mathbf{U}}_a \rightarrow \mathfrak{M}$  be the composition of the quotient map  $\tilde{\mathbf{U}}_a \rightarrow \tilde{\mathbf{U}}_a // \mathrm{Stab}(p)$  and the map  $\psi$  in (5.35). Let  $C \subset \tilde{\mathbf{U}}_a$  be the strict transform of the line  $\{V(f_{a,a} + tx_3^4) \mid t \in \mathbb{C}\}$ . Then  $\zeta(C) \subset W_1$ . Moreover, let  $\mathcal{U} \subset \tilde{\mathbf{U}}_a$  be a  $\mathrm{Stab}(p)$ -invariant open (in the classical topology) neighborhood of  $p$  such that the restriction of  $\psi$  to  $\mathcal{U} // \mathrm{Stab}(p)$  is an isomorphism onto  $\psi(\mathcal{U} // \mathrm{Stab}(p))$ ; then  $x \in \mathcal{U}$  is mapped to  $W_1$  by  $\zeta$  and has closed  $\mathrm{SL}_2$ -orbit if and only if  $x = V(f_{a,a} + tx_3^4)$  for some  $t \in \mathbb{C}$ .*

*Proof.* First  $(q + (a+u)x_3^2)(q + (a-u)x_3^2) = f_{a,a} - u^2x_3^4$  shows that  $\zeta(C) \subset W_1$ . For the remaining statement see the proof of **Claim 5.9**. □

5.4.4. *Moduli of K3 surfaces which are generic double cones.* Let  $\Lambda$  be the graded  $\mathbb{C}$ -algebra

$$(5.36) \quad \Lambda := \mathbf{S}^\bullet(V(4)^\vee \oplus V(6)^\vee \oplus V(8)^\vee),$$

where  $V(2d)^\vee$  has degree  $d$ . Then  $\mathrm{PSL}(2)$  acts on  $\mathrm{Proj} \Lambda$ , and  $\mathcal{O}_{\mathrm{Proj} \Lambda}(1)$  is naturally linearized. The involution

$$\begin{aligned} \mathrm{Proj} \Lambda &\longrightarrow \mathrm{Proj} \Lambda \\ [f, g, h] &\mapsto [f, -g, h] \end{aligned}$$

commutes with the action of  $\mathrm{PSL}(2)$ , and hence there is a (faithful) action of

$$(5.37) \quad G_c := \mathrm{PSL}(2) \times \mathbb{Z}/(2)$$

on  $\text{Proj } \Lambda$ . We let

$$(5.38) \quad \mathfrak{M}_c := \text{Proj } \Lambda // G_c$$

be the GIT quotient. We will show that  $\mathfrak{M}_c$  is naturally a compactification of the moduli space of hyperelliptic quartic  $K3$  surfaces which are double covers of a quadric cone with branch divisor not containing the vertex of the cone. First, we think of  $\text{SL}_2$  as the double cover of  $\text{SO}(q)$ , where  $q = x_0^2 + x_1^2 + x_2^2$  is as in **Subsubsection 5.4.2**, and correspondingly  $V(2d)$  is a subrepresentation of  $\mathbb{C}[x_0, x_1, x_2]_d$ . We associate to  $\xi := (f, g, h) \in V(4) \oplus V(6) \oplus V(8)$ , the quartic

$$(5.39) \quad B_\xi := V(x_3^4 + fx_3^2 + gx_3 + h).$$

Thus  $V(4) \oplus V(6) \oplus V(8)$  is identified with the set of such quartics. Both  $G_c$  and the multiplicative group  $\mathbb{C}^*$  act on the set of such quartics (the second group acts by rescaling  $x_3$ ). The quotient of  $(V(4) \oplus V(6) \oplus V(8)) \setminus \{0\}$  by the  $\mathbb{C}^*$  action is  $\text{Proj } \Lambda$ , hence  $\mathfrak{M}_c$  is identified with the quotient  $(V(4) \oplus V(6) \oplus V(8)) \setminus \{0\}$  by the full  $G_c \times \mathbb{C}^*$ -action. Given  $[\xi] \in \text{Proj } \Lambda$ , we let  $X_\xi$  be the double cover of the cone  $V(q) \subset \mathbb{P}_{\mathbb{C}}^3$  ramified over the restriction of  $B_\xi$  to  $V(q)$ , and  $L_\xi$  be the degree-4 polarization of  $X_\xi$  pulled back from  $\mathcal{O}_{\mathbb{P}^3}(1)$ .

**Proposition 5.15.** *Let  $[\xi] \in \text{Proj } \Lambda$  be such that  $X_\xi$  has rational singularities. Then  $[\xi]$  is  $G_c$ -stable. The open dense subset of  $\mathfrak{M}_c$  parametrizing isomorphism classes of such  $[\xi]$  is the moduli space of polarized quartics which are double covers of a quadric cone with branch divisor not containing the vertex of the cone.*

*Proof.* Let  $[\xi] = [f, g, h] \in \text{Proj } \Lambda$  be a non-stable point. Then by the Hilbert-Mumford Criterion there exist a point  $a \in \mathbb{P}^1$  (where  $\mathbb{P}^1$  is identified with the conic  $V(q, x_3)$  via the Veronese embedding) such that

$$(5.40) \quad \text{mult}_a(f) \geq 2, \quad \text{mult}_a(g) \geq 3, \quad \text{mult}_a(h) \geq 4.$$

The point  $a \in \mathbb{P}^1$  is identified with a point  $p \in V(q, x_3)$  (as recalled above), which belongs to the quartic  $B_\xi$ . The inequalities in (5.40) give that the multiplicity at  $p$  of the divisor  $B_\xi|_{V(q)}$  is at least 4, and hence the corresponding double cover of  $V(q)$  (i.e.  $X_\xi$ ) does *not* have rational singularities. This proves the first statement. The rest of the proof is analogous to Shah's proof (see Theorem 4.3 in [Sha80]) that  $\mathfrak{M}_u$  (see (5.6)) is a compactification of the moduli space for unigonal  $K3$  surfaces. The key point is that any quartic not containing the vertex  $[0, 0, 0, 1]$  has such an equation after a suitable projectivity  $\varphi$  (a Tschirnhaus transformation) of the form  $\varphi^*x_i = x_i$ ,  $\varphi^*x_3 = x_3 + \ell(x_0, x_1, x_2)$  where  $\ell(x_0, x_1, x_2)$  is homogeneous of degree 1.  $\square$

Let  $[\xi] \in \text{Proj } \Lambda$  be generic; then  $(X_\xi, L_\xi)$  is a polarized quartic  $K3$  surface whose period point belongs to  $H_h^{(2)}$ , which (see [LO16]) is identified with  $\mathcal{F}(17)$  via the embedding  $f_{17,19}: \mathcal{F}(17) \hookrightarrow \mathcal{F}$ . Thus we have a rational period map

$$(5.41) \quad \mathbf{p}_c: \mathfrak{M}_c \dashrightarrow \mathcal{F}(17)^* \subset \mathcal{F}^*.$$

A generic polarized quartic  $K3$  surface is a double cover of the quadric cone unramified over the vertex, and hence is isomorphic to  $(X_\xi, L_\xi)$  for a certain  $[\xi] \in \text{Proj } \Lambda$ . By the global Torelli Theorem for  $K3$  surfaces, it follows that the period map  $\mathbf{p}_c$  is birational.

**5.4.5. Partial extension of the period map on a weighted blow-up: the case of a point in  $\widetilde{W}_1 \setminus E_\omega$ .** Let  $(a, b) \in \mathbb{C}^2$ , with  $a \neq b$ . Let  $N_{a,b}$  be the  $\text{SL}_2$  representation in (5.26), and let  $M_{a,b}$  be the sub-representation

$$(5.42) \quad M_{a,b} := V(8) \oplus V(6) \oplus R \cdot x_3^2.$$

Let  $\mathbf{N}_{a,b}$  be the normal slice of  $V(f_{a,b})$  defined in **Subsubsection 5.4.2**, and let  $\mathbf{M}_{a,b} \subset \mathbf{N}_{a,b}$  be the subspace

$$\mathbf{M}_{a,b} := \{V(f_{a,b} + g) \mid g \in M_{a,b}\}.$$

Notice that

$$\dim \mathbf{M}_{a,b} = 21.$$

Let  $(z_1, \dots, z_5)$  be coordinates on  $V(4)$ , let  $(z_6, \dots, z_{12})$  be coordinates on  $V(6)$ , and let  $(z_{13}, \dots, z_{21})$  be coordinates on  $V(8)$ ; thus  $(z_1, \dots, z_{21})$  are coordinates on  $\mathbf{M}_{a,b}$  (with a slight abuse of notation) centered at  $V(f_{a,b})$ . Let  $\sigma$  be the weight defined by

$$(5.43) \quad \sigma(z_i) := \begin{cases} 2 & \text{if } i \in \{1, 5\}, \\ 3 & \text{if } i \in \{6, 12\}, \\ 4 & \text{if } i \in \{13, 21\}. \end{cases}$$

Let  $\widehat{\mathbf{M}}_{a,b} := \text{Bl}_\sigma(\mathbf{M}_{a,b})$  be the corresponding weighted blow up, and let  $E_{a,b}$  be the exceptional set of  $\widehat{\mathbf{M}}_{a,b} \rightarrow \mathbf{M}_{a,b}$ . Thus  $E_{a,b}$  is the weighted projective space  $\mathbb{P}(2^5, 3^7, 4^9) \cong \text{Proj } \Lambda$ , where  $\Lambda$  is the graded ring in (5.36) (with grading defined right after (5.36)). The action of  $\text{Aut}(V_{f_{a,b}}) = G_c$  (here  $G_c$  is as in (5.37)) on  $\mathbf{M}_{a,b}$  lifts to an action on  $\widehat{\mathbf{M}}_{a,b}$ . Thus there is an associated GIT quotients  $\widehat{\mathbf{M}}_{a,b} // G_c$ . The map  $\widehat{\mathbf{M}}_{a,b} \rightarrow \mathbf{M}_{a,b}$  induces a map

$$(5.44) \quad \widehat{\theta}: \widehat{\mathbf{M}}_{a,b} // G_c \longrightarrow \mathbf{M}_{a,b} // G_c.$$

Moreover, we have the set-theoretic equality

$$(5.45) \quad \widehat{\theta}^{-1}(\overline{V(f_{a,b})})_{\text{red}} = \text{Proj } \Lambda // G_c = \mathfrak{M}_c.$$

Since the natural map  $\mathbf{M}_{a,b} // G_c \rightarrow \mathfrak{M}$  is dominant, it makes sense to compose it with the (rational) period map  $\mathbf{p}: \mathfrak{M} \dashrightarrow \mathcal{F}^*$ . Composing with the birational map in (5.44), we get a rational map

$$(5.46) \quad \widehat{\mathbf{p}}_{a,b}: \widehat{\mathbf{M}}_{a,b} // G_c \dashrightarrow \mathcal{F}^*.$$

**Proposition 5.16.** *With notation as above, the restriction of  $\widehat{\mathbf{p}}_{a,b}$  to  $\widehat{\theta}^{-1}(\overline{V(f_{a,b})})_{\text{red}} = \mathfrak{M}_c$  is equal to the composition of the automorphism*

$$(5.47) \quad \begin{array}{ccc} \mathfrak{M}_c & \xrightarrow{\varphi_{a,b}} & \mathfrak{M}_c \\ [f, g, h] & \mapsto & [f, -\frac{i}{2}(a-b)g, -\frac{1}{4}(a-b)^2h] \end{array}$$

and the period map in (5.41). Moreover  $\widehat{\mathbf{p}}_{a,b}$  is regular at all points of  $\widehat{\theta}^{-1}(\overline{V(f_{a,b})})_{\text{red}}$  where  $\mathbf{p}_c$  is regular.

*Proof.* Let  $[\xi] = [f, g, h] \in \text{Proj } \Lambda = E_{a,b}$  be a  $G_c$ -semistable point with corresponding point  $\overline{[\xi]} \in \mathfrak{M}_c$ , and let  $[\eta] = \varphi_{a,b}(\overline{[\xi]})$ . Suppose that the period map  $\mathbf{p}_c$  is regular at  $[\eta]$ . We will prove that if  $\Delta \subset \mathbb{C}$  is a disc centered at 0, and  $\Delta \rightarrow \widehat{\mathbf{M}}_{a,b}$  is an analytic map mapping 0 to  $[\xi]$  and no other point to the exceptional divisor  $E_{a,b}$ , then the period map is defined on a neighborhood of  $0 \in \Delta$ , and its value at 0 is equal to the period point of  $(X_\eta, L_\eta)$ . This will prove the Theorem, by **Corollary 5.2**. By **Proposition 5.1** the statement that we just gave boils down to the following computation. First, we identify  $V(2d)$  with the corresponding  $\text{SO}(q)$ -sub-representation of  $\mathbb{C}[x_0, x_1, x_2]_d$ ; thus  $f, g, h \in \mathbb{C}[x_0, x_1, x_2]$  are homogeneous of degrees 2, 3 and 4 respectively. Now let  $\mathcal{X} \subset \mathbb{P}^3 \times \Delta$  be the hypersurface given by the equation

$$(5.48) \quad 0 = (q + ax_3^2)(q + bx_3^2) + t^2x_3^2(f + tF) + t^3x_3(g + tG) + t^4(h + tH)$$



where  $F \in \mathbb{C}[x_0, x_1, x_2]_2[[t]]$ ,  $G \in \mathbb{C}[x_0, x_1, x_2]_3[[t]]$ , and  $H \in \mathbb{C}[x_0, x_1, x_2]_4[[t]]$ . Now consider the 1-parameter subgroup of  $GL_4(\mathbb{C})$  defined by  $\lambda(t) := \text{diag}(1, 1, 1, t)$ . We let  $\mathcal{Y} \subset \mathbb{P}^3 \times \Delta$  be the closure of

$$\{([x], t) \mid t \neq 0, \quad \lambda(t)[x] \in \mathcal{X}\}.$$

Then  $Y_t \cong X_t$  for  $t \neq 0$ , and  $\mathcal{Y}$  has equation

$$(5.49) \quad 0 = q^2 + t^2(a + b)x_3^2q + t^4(abx_3^4 + x_3^2f + x_3g + h) + t^5(\dots)$$

Let  $\nu: \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  be the normalization of  $\mathcal{Y}$ . Dividing (5.49) by  $t^4x_i^4$ , we get that the ring of regular functions of the affine set  $\nu^{-1}(\mathcal{Y} \cap \mathbb{P}_{x_i}^3)$  is generated over  $\mathbb{C}[\mathcal{Y} \cap \mathbb{P}_{x_i}^3]$  by the rational function  $\xi_i := q/(x_i^2t^2)$ , which satisfies the equation

$$0 = \xi_i^2 + (a + b) \left( \frac{x_3}{x_i} \right)^2 \xi_i + (abx_3^4 + x_3^2f + x_3g + h)/x_i^4 + t(\dots)$$

It follows that for  $t \rightarrow 0$  the quartics  $X_t$  approach the double cover of  $V(q)$  branched over the intersection with the quartic

$$(5.50) \quad 0 = ((a + b)x_3^2)^2 - 4(abx_3^4 + x_3^2f + x_3g + h) = (a - b)^2x_3^4 - 4x_3^2f - 4x_3g - 4h.$$

□

5.4.6. *Partial extension of the period map on a weighted blow-up: the unique point in  $\widetilde{W}_1 \cap E_\omega$ . Let  $a \neq 0$ , and*

$$\widetilde{\mathbf{V}}_a := \widetilde{\mathbf{U}}_a \cap E_{\mathbf{D}}.$$

(We recall that  $E_{\mathbf{D}}$  is the exceptional divisor of the blow-up  $P \rightarrow |\mathcal{O}_{\mathbb{P}^3}(4)|$  with center the closed subset  $\mathbf{D}$  parametrizing double quadrics.) Thus, letting  $S$  be as in (5.30), we have

$$(5.51) \quad \widetilde{\mathbf{V}}_a = \mathbb{P}(S) = \mathbb{P}(V(8) \oplus V(6) \oplus R \cdot x_3^2 \oplus \langle x_3^4 \rangle), \quad \dim \widetilde{\mathbf{V}}_a = 21.$$

Let  $p := (Q_a, x_3^4) \in \widetilde{\mathbf{V}}_a$ , see **Notation/Convention 5.10**. Then  $\widetilde{\mathbf{V}}_a$  is mapped to itself by  $\text{Stab}(p)$ , and by restriction of the map  $\psi$  in (5.35) we get a map

$$\widetilde{\mathbf{V}}_a // \text{Stab}(p) \longrightarrow \widetilde{\mathfrak{M}}.$$

We define a weighted blow up of  $\widetilde{\mathbf{V}}_a$  with center  $p$  as follows. First, by (5.51) we have the following description of an affine neighborhood  $T$  of  $p \in \widetilde{\mathbf{V}}_a$ :

$$\begin{array}{ccc} V(8) \oplus V(6) \oplus R \cdot x_3^2 & \longrightarrow & T \\ \alpha & \mapsto & [x_3^4 + \alpha] \end{array}$$

Let  $(z_1, \dots, z_5)$  be coordinates on  $R \cdot x_3^2 = V(4)$ , let  $(z_6, \dots, z_{12})$  be coordinates on  $V(6)$ , and let  $(z_{13}, \dots, z_{21})$  be coordinates on  $V(8)$ ; thus  $(z_1, \dots, z_{21})$  are coordinates on  $T$  (with a slight abuse of notation) centered at the point  $p$ . Let  $\sigma$  be the weight defined by

$$(5.52) \quad \sigma(z_i) := \begin{cases} 2 & \text{if } i \in \{1, 5\}, \\ 3 & \text{if } i \in \{6, 12\}, \\ 4 & \text{if } i \in \{13, 21\}. \end{cases}$$

(Note: we are proceeding exactly as in **Subsubsection 5.4.5**.) Let  $\widehat{\mathbf{V}}_a := \text{Bl}_\sigma(\widetilde{\mathbf{V}}_a)$  be the corresponding weighted blow up, and let  $E_a$  be the corresponding exceptional divisor. Thus  $E_a$  is the weighted projective space  $\mathbb{P}(2^5, 3^7, 4^9) \cong \text{Proj } \Lambda$ , where  $\Lambda$  is the graded ring in (5.36) (with grading defined right after (5.36)). The action of  $\text{Aut}(p)$  on  $\widetilde{\mathbf{V}}_a$  lifts to an action on  $\widehat{\mathbf{V}}_a$ . There is an associated GIT quotient  $\widehat{\mathbf{V}}_a // \text{Stab}(p)$ , and a regular map

$$\widehat{\eta}: \widehat{\mathbf{V}}_a // \text{Stab}(p) \longrightarrow \widetilde{\mathbf{V}}_a // \text{Stab}(p).$$

We have the set-theoretic equality

$$(5.53) \quad \widehat{\eta}^{-1}(\overline{p})_{\text{red}} = \text{Proj } \Lambda // G_c = \mathfrak{M}_c.$$

We have a rational map

$$(5.54) \quad \widehat{\mathbf{p}}_a: \widehat{\mathbf{V}}_a // \text{Aut}(p) \dashrightarrow \mathcal{F}^*.$$

**Proposition 5.17.** *With notation as above, the restriction of  $\widehat{\mathbf{p}}_a$  to  $\widehat{\eta}^{-1}(\overline{p})_{\text{red}} = \mathfrak{M}_c$  is equal to the period map in (5.41). Moreover  $\widehat{\mathbf{p}}_a$  is regular at all points of  $\widehat{\eta}^{-1}(\overline{p})_{\text{red}}$  where  $\mathbf{p}_c$  is regular.*

*Proof.* Let  $[\xi] = [f, g, h] \in \text{Proj } \Lambda = E_a$  be a  $G_c$ -semistable point with corresponding point  $\eta \in \mathfrak{M}_c$ . Suppose that the period map  $\mathbf{p}_c$  is regular at  $\eta$ . We will prove that if  $\Delta \subset \mathbb{C}$  is a disc centered at 0, and  $\Delta \rightarrow \widehat{\mathbf{V}}_a$  is an analytic map mapping 0 to  $[\xi]$  and no other point to the exceptional divisor  $E_a$ , then the period map is defined on a neighborhood of  $0 \in \Delta$ , and its value at 0 is equal to the period point of  $(X_\eta, L_\eta)$ . This will prove the Theorem, by **Corollary 5.2**. By **Proposition 5.1**, the previous statement boils down to the following computation. Let  $f, g, h \in \mathbb{C}[x_0, x_1, x_2]$  be homogeneous of degrees 2, 3 and 4 respectively, not all zero. Let  $C_t \subset V(q + ax_3^2)$  be the intersection with the quartic

$$x_3^4 + t^2 x_3^2(f + tF) + t^3 x_3(g + tG) + t^4(h + tH) = 0.$$

where  $F \in \mathbb{C}[x_0, x_1, x_2]_2[[t]]$ ,  $G \in \mathbb{C}[x_0, x_1, x_2]_3[[t]]$ , and  $H \in \mathbb{C}[x_0, x_1, x_2]_4[[t]]$ . We will show that  $C_t$  for  $t \neq 0$  approaches for  $t \rightarrow 0$ , the curve

$$q = x_3^4 + x_3^2 f + x_3 g + h = 0.$$

In fact it suffices to consider the limit for  $t \rightarrow 0$  of  $\lambda(t)C_t$ , where  $\lambda$  is the 1-PS  $\lambda(t) = (1, 1, 1, t)$ .  $\square$

**5.4.7. A global modification of  $\widetilde{\mathfrak{M}}$  and partial extension of the period map.** Let  $\mathbf{T} \subset |\mathcal{O}_{\mathbb{P}^3}(4)|$  be the closure of the set of  $\text{PGL}(4)(\mathbb{C})$ -translates of  $V(f_{a,b})$ , for all  $(a, b) \in \mathbb{C}^2$ . Thus  $\mathbf{T}$  is a closed,  $\text{PGL}(4)(\mathbb{C})$ -invariant subset, containing  $\mathbf{D}$  (the set of double quadrics), and

$$(5.55) \quad \dim \mathbf{T} = 13.$$

Let  $\widetilde{\mathbf{T}} \subset P$  be the strict transform of  $\mathbf{T}$  in the blow-up  $\pi: P \rightarrow |\mathcal{O}_{\mathbb{P}^3}(4)|$  with center  $\mathbf{D}$ . The set of semistable points  $\widetilde{\mathbf{T}}^{ss} \subset \widetilde{\mathbf{T}}$  (for a polarization  $\pi^* \mathcal{L}(-\epsilon E_{\mathbf{D}})$  close to  $\pi^* \mathcal{L}$ , see the proof of **Proposition 5.5**) is the union of the set of points of  $\widetilde{\mathbf{T}} \setminus E_{\mathbf{D}}$  which are mapped by  $\pi$  to quartics  $\text{PGL}(4)(\mathbb{C})$ -equivalent to  $V(f_{a,b})$  for some  $a \neq b$ , and of  $\widetilde{\mathbf{T}} \cap E_{\mathbf{D}}^{ss}$ . The latter set consists of the  $\text{PGL}(4)(\mathbb{C})$ -translates of the points  $(Q_a, x_3^4)$  defined in **Notation/Convention 5.10**.

In **Subsubsection 5.4.5** and **Subsubsection 5.4.6** we defined a weighted blow up of an explicit normal slice to  $\widetilde{\mathbf{T}}$  at points  $x \in \widetilde{\mathbf{T}}^{ss}$ . That construction can be globalized: one obtains a modification  $\widehat{\pi}: \widehat{P} \rightarrow P$  which is an isomorphism away from  $P \setminus \widetilde{\mathbf{T}}$ , and replaces  $\widetilde{\mathbf{T}}^{ss}$  by a locally trivial fiber bundle over  $\widetilde{\mathbf{T}}^{ss}$  with fiber isomorphic to the weighted projective space  $\mathbb{P}(2^5, 3^7, 4^9)$ . In fact the weighted blow up is isomorphic to the usual blow up of a suitable ideal, see Remark 2.5 of [And16], hence one may define an ideal  $\mathcal{I}$  co-supported on  $\widetilde{\mathbf{T}}$  such that  $\widehat{P} = \text{Bl}_{\mathcal{I}} P$ .

Let  $E_{\widetilde{\mathbf{T}}}$  be the exceptional divisor of  $\widehat{\pi}$ . Letting  $\mathcal{L}_P := \pi^* \mathcal{L}(-\epsilon E_{\mathbf{D}})$  be a polarization of  $P$  as above, we may consider the GIT quotient of  $\widehat{P}$  with  $\text{PGL}(4)(\mathbb{C})$ -linearized polarization  $\mathcal{L}_{\widehat{P}} := \pi^* \mathcal{L}_P(-t E_{\widetilde{\mathbf{T}}})$ , call it  $\widehat{\mathfrak{M}}(t)$ . For  $0 < t$  small enough, the map  $\widehat{\pi}$  induces a regular map  $\widehat{\mathfrak{M}}(t) \rightarrow \widetilde{\mathfrak{M}}$ . From now on we drop the parameter  $t$  from our notation; thus  $\widehat{\mathfrak{M}}$  denotes  $\widehat{\mathfrak{M}}$  for  $t$  small.

The image of  $E_{\widetilde{\mathbf{T}}}$  in  $\widehat{\mathfrak{M}}$  is a fiber bundle

$$\rho: E_1 \rightarrow \widetilde{W}_1,$$

with fiber  $\mathfrak{M}_c$  over every point. Let  $\widehat{\mathbf{p}}: \widehat{\mathfrak{M}} \dashrightarrow \mathcal{F}^*$  be the period map. We claim that the restriction of  $\widehat{\mathbf{p}}$  to the fiber of  $E_1 \rightarrow \widetilde{W}_1$  over  $x$  is regular away from the indeterminacy locus of  $\mathbf{p}_c: \mathfrak{M}_c \dashrightarrow \mathcal{F}^*$ ,

and it has the same value, provided we compose with the automorphism of  $\mathfrak{M}_c$  given by (5.47) if  $x \notin E_\omega$  and  $\hat{\pi}(x) = [V(f_{a,b})]$ .

In order to prove the claim it suffices to prove the following. Let  $\Delta \subset \mathbb{C}$  be a disc centered at 0, and let  $\Delta \rightarrow \widehat{\mathfrak{M}}$  be an analytic map mapping 0 to a point  $\hat{x} \in E_1$  such that the period map  $\mathfrak{p}_c$  is regular at the point  $\eta \in \mathfrak{M}_c = \rho^{-1}(\rho(\hat{x}))$  corresponding to  $\hat{x}$ , and suppose that  $(\Delta \setminus \{0\})$  is mapped to the complement of  $E_1$  and into the locus where the period map is regular; then the value at 0 of the extension of the period map on  $\Delta \setminus \{0\}$  is equal to the period point of  $(X_\eta, L_\eta)$ . We may assume that  $\Delta \rightarrow \widehat{\mathfrak{M}}$  lifts to an analytic map  $\tau: \Delta \rightarrow \hat{P}$  mapping 0 to a point of  $E_{\tilde{T}}$  with closed orbit (in the semistable locus) lifting  $\hat{x}$ . In **Subsubsection 5.4.5** and **Subsubsection 5.4.6** we have checked that the value at 0 of the extension behaves as required if  $\hat{\pi} \circ \tau(\Delta)$  is contained in the normal slice to  $\tilde{T}$  at the point  $\hat{\pi} \circ \tau(0)$  (defined in **Subsubsection 5.4.5** and **Subsubsection 5.4.6** respectively).

It remains to prove that it behaves as required also if the latter condition does not hold. If  $\hat{\pi} \circ \tau(0) \notin E_{\mathbf{D}}$ , then the argument is similar to that given in **Subsubsection 5.4.5**; one simply replaces  $a, b \in \mathbb{C}$  by holomorphic functions  $a(t), b(t)$  where  $t \in \Delta$ .

If  $\hat{\pi} \circ \tau(0) \in E_{\mathbf{D}}$ , one needs a separate argument. The relevant computation goes as follows. Let  $f, g, h \in \mathbb{C}[x_0, x_1, x_2]$  be homogeneous of degrees 2, 3 and 4 respectively, not all zero. Let  $\mathcal{X} \subset \mathbb{P}^3 \times \Delta$  be the hypersurface given by the equation

$$(5.56) \quad (q + x_3^2)^2 + t^{4k} x_3^4 + t^{4k+6p} x_3^2(f + tF) + t^{4k+9p} x_3(g + tG) + t^{4k+12p}(h + tH) = 0,$$

where

$$F \in \mathbb{C}[x_0, x_1, x_2]_2[[t]], \quad G \in \mathbb{C}[x_0, x_1, x_2]_3[[t]], \quad H \in \mathbb{C}[x_0, x_1, x_2]_4[[t]].$$

Let  $\lambda(t) := \text{diag}(1, 1, 1, t^4)$ , and let  $\mathcal{Y} \subset \mathbb{P}^3 \times \Delta$  be the closure of

$$\{([x], t) \mid t \neq 0, \quad \lambda(t)[x] \in \mathcal{X}\}.$$

Thus  $Y_t \cong X_t$  for  $t \neq 0$ , and  $\mathcal{Y}$  has equation

$$(5.57) \quad q^2 + 2t^8 q x_3^2 + t^{16} x_3^4 + t^{4k+16} x_3^4 + t^{4k+6p+8} x_3^2(f + tF) + t^{4k+9p+4} x_3(g + tG) + t^{4k+12p}(h + tH) = 0.$$

Dividing the above equation by  $t^{16}$  we find that the rational function  $\xi_i := q/(x_i^2 t^8)$  satisfies the equation

$$0 = \xi_i^2 + 2 \left( \frac{x_3}{x_i} \right)^2 \xi_i + (x_3^4 + t^{4k} x_3^4 + t^{4k+6p-8} x_3^2(f + tF) + t^{4k+9p-12} x_3(g + tG) + t^{4k+12p-16}(h + tH))/x_i^4.$$

It follows that the fiber at  $t = 0$  of the normalization of  $\mathcal{Y}$  is the double cover of  $V(q)$  ramified over the intersection with the limit for  $t \rightarrow 0$  of the quartic

$$4x_3^4 - 4(x_3^4 + t^{4k} x_3^4 + t^{4k+6p-8} x_3^2(f + tF) + t^{4k+9p-12} x_3(g + tG) + t^{4k+12p-16}(h + tH)) = 0.$$

Replacing  $x_3$  by  $t^{-3p+4} x_3$  we get that the fiber at  $t = 0$  of the normalization of  $\mathcal{Y}$  is the double cover of  $V(q)$  ramified over the intersection with the quartic

$$(5.58) \quad x_3^4 + x_3^2 f + x_3 g + h = 0.$$

Let us explain why the above computation proves the required statement. Let  $\epsilon: \Delta \rightarrow |\mathcal{O}_{\mathbb{P}^3}(4)|$  be the analytic map defined by  $\epsilon(t) := X_t$ . Then  $\text{Im}(\epsilon) \subset S_1$ , where  $S_1$  is as in (5.31) (notice that  $X_0 = f_{1,1}$ ). Let  $\tilde{U}_1$  be the blow up of  $S_1 \setminus Y_1$  with center  $V(f_{1,1})$ , see **Subsubsection 5.4.3**, and let  $\tilde{\epsilon}: \Delta \rightarrow \tilde{U}_1$  be the lift of  $\epsilon$  (by shrinking  $\Delta$  we may assume that  $\text{Im}(\epsilon) \cap Y_1 = \emptyset$ ). Then  $\tilde{\epsilon}(0) = p = (Q_1, x_3^4)$ , notation as in **Notation/Convention 5.10**.

Now, choose a basis  $\{a_0, \dots, a_{21}\}$  of the  $\text{SL}_2$ -representation  $S$  given by (5.30) adapted to the decomposition in (5.30); more precisely  $a_0 = x_3^4$ ,  $\{a_1, \dots, a_5\}$  is a basis of  $R \cdot x_3^2$ ,  $\{a_6, \dots, a_{12}\}$  is a basis of  $V(6)$ , and  $\{a_9, \dots, a_{21}\}$  is a basis of  $V(8)$ . Let  $\{w_0, \dots, w_{21}\}$  be the basis dual to  $\{a_0, \dots, a_{21}\}$ ; then  $(w_0, \dots, w_{21})$  are coordinates on an affine neighborhood of  $V(f_{a,a})$  in  $\mathbf{S}_a$ , centered at  $V(f_{a,a})$ .

Next set  $y_0 = w_0$ , and  $y_i = w_i/w_0$  for  $i \in \{1, \dots, 22\}$ . Then  $(y_0, \dots, y_{21})$  are coordinates on an affine neighborhood of  $p \in \widetilde{U}_1$ , centered at  $p$ . Let  $(z_1, \dots, z_{21})$  be the affine coordinates introduced in **Subsubsection 5.4.6**; we may assume that  $y_i|_{\widetilde{V}_1} = z_i$  for  $i \in \{1, \dots, 22\}$ .

In the coordinates  $(y_0, \dots, y_{21})$  we have

$$\tilde{c}(t) = (t^{4k}, t^{6p}(f_1 + tF_1), \dots, t^{6p}(f_5 + tF_5), t^{9p}(g_5 + tG_5), \dots, t^{9p}(g_{12} + tG_{12}), t^{12p}(h_{13} + tH_{13}), \dots, t^{12p}(h_{21} + tH_{21})),$$

with obvious notation:  $(f_1, \dots, f_5)$  are the coordinates of  $f$  in the basis  $\{a_1, \dots, a_5\}$ , etc. The computation above shows that the extension at 0 of the period map is equal to the period point of the double cover of  $V(q)$  ramified over the intersection with the quartic defined by (5.58), and hence the period map is regular at the point corresponding to  $[f, g, h]$  by **Proposition 5.1** and **Corollary 5.2**.

5.4.8. *The first flip and a contraction of  $\widehat{\mathfrak{M}}$ .* The divisor  $E_1 \subset \widehat{\mathfrak{M}}$  is isomorphic to  $\widetilde{W}_1 \times \mathfrak{M}_c$ . The normal bundle of  $E_1$  restricted to the fibers of the projection  $E_1 \rightarrow \mathfrak{M}_c$  is negative; it follows that (in the analytic category) there exists a contraction  $\widehat{\mathfrak{M}} \rightarrow \mathfrak{M}_{1/2}$  of  $E_1$  along the the fibers of  $E_1 \rightarrow \mathfrak{M}_c$ . We claim that  $\mathfrak{M}_{1/2}$  must be isomorphic to  $\mathcal{F}(1/3, 1/2)$ . In fact, let  $\widehat{p}: \widehat{\mathfrak{M}} \dashrightarrow \mathcal{F}$  be the period map (notice: contrary to previous notation, the codomain is  $\mathcal{F}$ , not  $\mathcal{F}^*$ ). The generic fiber of  $E_1 \rightarrow \mathfrak{M}_c$  is in the regular locus of  $\widehat{p}$ , and is mapped to a constant: it follows that

$$(5.59) \quad 0 = \widehat{p}^*(\lambda) \cdot (\widetilde{W}_1 \times \{[f, g, h]\}) = \widehat{p}^*(\Delta) \cdot (\widetilde{W}_1 \times \{[f, g, h]\}), \quad [f, g, h] \in \mathfrak{M}_c.$$

On the other hand, letting  $p \in \widetilde{W}_1$ , and adopting the notation of [LO16], we have

$$(5.60) \quad \widehat{p}(\{p\} \times \mathfrak{M}_{c,reg}) \subset \text{Im}(f_{17,19}).$$

(Here  $\mathfrak{M}_{c,reg}$  is the set of regular points of the period map  $\widehat{p}_c: \mathfrak{M}_c \dashrightarrow \mathcal{F}$ ; by **Proposition 5.17** it is equal to the intersection of  $\{p\} \times \mathfrak{M}_{c,reg}$  with the set of regular points of  $\widehat{p}$ .) By Proposition 5.3.7 of [LO16] we have  $f_{17,19}^*(\lambda + \beta\Delta) = (1 - 2\beta)\lambda(17) + \beta\Delta(17)$ . Now,  $\Delta(17) = H_h(17)/2$ , and  $\widehat{p}(\{p\} \times \mathfrak{M}_c)$  avoids the support of  $H_h(17) = \text{Im } f_{16,17}$  by **Proposition B.1**. Thus

$$(5.61) \quad \widehat{p}^*(\lambda + \beta\Delta)|_{\{p\} \times \mathfrak{M}_c} = \widehat{p}^*((1 - 2\beta)\lambda)|_{\{p\} \times \mathfrak{M}_c}.$$

The conclusion is that  $\widehat{p}^*(\lambda + \beta\Delta)$  contracts all of  $E_1$  to a point if  $\beta \geq 1/2$  (and is trivial on  $E_1$  if  $\beta = 1/2$ ), while if  $\beta < 1/2$ , then the restriction of  $\widehat{p}^*(\lambda + \beta\Delta)$  to  $E_1$  is the pull-back of an ample line bundle on  $\mathfrak{M}_c$ . Thus we expect that for  $\beta < 1/2$  close to  $1/2$  the  $(\mathbb{Q})$  line-bundle  $\widehat{p}^*(\lambda + \beta\Delta)$  is the pull-back of an ample  $(\mathbb{Q})$  line bundle on  $\mathfrak{M}_{1/2}$ , and hence  $\mathfrak{M}_{1/2}$  is identified with  $\mathcal{F}(\beta)$ , because the period map would be birational map which is an isomorphism in codimension 2 and pulls- back an ample line bundle to an ample line bundle.

**5.5. Semistable reduction for Dolgachev singularities, and the exceptional flips.** In this subsection, we will give some evidence relating the exceptional flips (i.e. those occurring for  $\beta \in \{\frac{1}{9}, \frac{1}{7}, \frac{1}{6}\}$ ) to the loci of quartics with  $E_{12}$ ,  $E_{13}$ , and  $E_{14}$  singularities respectively. The behavior here is very similar to the first steps in the Hassett-Keel program. Specifically, we recall that in the variation of log canonical models  $\mathcal{M}_g(\alpha) = \text{Proj}(\overline{\mathcal{M}}_g, K_{\overline{\mathcal{M}}_g} + \alpha\Delta_{\overline{\mathcal{M}}_g})$  (for  $\alpha \in [0, 1]$ ) for the moduli space of genus  $g$  curves  $\overline{\mathcal{M}}_g$ , the first critical value is  $\alpha = \frac{9}{11}$  which corresponds to replacing the curves with elliptic tails by cuspidal curves. Similarly, at the next critical value  $\alpha = \frac{7}{10}$ , the locus of curves with elliptic bridges is replaced by the locus of curves with tacnodes (see [HH09, HH13] for details). In our situation, the singularities  $E_{12}$ ,  $E_{13}$ , and  $E_{14}$  are analogous to cusps and tacnodes, i.e. the simplest non-log canonical singularities in dimension 2. While the analogue of elliptic tail and bridge is that of marked  $K3$  surface (in the sense Dolgachev's lattice polarized  $K3$ ).

5.5.1. *KSBA (semi)stable replacement.* According to the general KSBA philosophy, for varieties of general type there exists a canonical compactification obtained by allowing degenerations with semi-log-canonical (slc) singularities and ample canonical bundle. In particular, any 1-parameter degeneration has a canonical limit with slc singularities. However, when studying GIT one ends up with compactifications that allows non-slc singularities. For example, the GIT compactification for quartic curves will allow quartics with cusp singularities. Thus a natural question is: *given a degeneration  $\mathcal{X}/\Delta$  of varieties of general type such that the general fiber is smooth (or mildly singular), but such that  $X_0$  does not have slc singularities, to find a stable KSBA replacement  $X'_0$*  (by MMP such  $X'_0$  exists and it is unique). Of course,  $X'_0$  depends on the original fiber  $X_0$  and on the family  $\mathcal{X}/\Delta$  (i.e. the choice of approaching  $X_0$ ). Motivated by Hassett-Keel program (many of the  $\mathcal{M}_g(\alpha)$  are obtained via GIT, and one has to compare them to  $\overline{\mathcal{M}}_g$ ), Hassett [Has00] has studied the influence of certain classes of curve singularities on the KSBA (semi)stable replacement (in this case, the usual nodal curve replacement). Hassett perspective is to consider  $C_0$  a curve with a unique non-slc (i.e. non-nodal) singularity, and to consider a generic smoothing of it  $\mathcal{C}/\Delta$ . The question is what can be said about the semi-stable replacement  $C'_0$  of  $C_0$ . Of course, one component of  $C'_0$  will be the normalization of  $C_0$  (assuming that this normalization is not a rational curve). The remaining components (and the gluing to  $\tilde{C}_0$ ) of  $C'_0$  (the “tail part”) will depend on the non-slc singularity of  $C_0$  and its smoothing, which is a local computation. The classical example is the semi-stable replacement for cuspidal curves (see [HM98, §3.C]), that we briefly review below.

**Example 5.18** (Semi-stable replacement for cuspidal curves). Locally a curve with a cusp has the equation  $y^2 + x^3$ , and a generic 1-parameter smoothing will be given by  $\mathcal{C} = V(t + y^2 + x^3) \rightarrow \Delta_t$ . After a base change of order 6, which is necessary to make the local monodromy action unipotent, one obtains a surface  $V(t^6 + y^2 + x^3) \subset (\mathbb{C}^3, 0)$  with a simple elliptic singularity at the origin. The weighted blow-up of the origin will resolve this singularity, and the resulting exceptional curve  $E$  is an elliptic curve (explicitly it is  $V(t^6 + y^2 + x^3) \subset \mathbb{WP}(1, 3, 2)$ ). The new family  $\mathcal{C}'$  (obtained by base change and weighted blow-up) will be a semi-stable family of curves, with the new central fiber consisting of the union of the normalization of  $C_0$  and of the exceptional curve  $E$  (“the elliptic tail”) glued at a single point. Note that instead of a weighted blow-up, one can use several regular blow-ups, these will lead first to a semi-stable curve with additional rational tails, which can be then contracted to give the stable model (with a single elliptic tail). The two blow-up (and then blow-down) processes are equivalent; the weighted blow-up has the advantage of being minimal, and generalizing well in our situation.

As mentioned above, Hassett [Has00] has generalized this for certain types of planar curve singularities (essentially weighted homogeneous, and related). In higher dimension (e.g. surfaces), much less is known - there is a similar computation (for surfaces with triangle singularities) to the elliptic curve example contained in an unpublished letter of Shepherd-Barron to Friedman (in connection to [Fri83] – such examples tend to give degenerations with finite, or even trivial, monodromy). Some similar computations appear in Gallardo [Gal13], and what is needed for our purposes will be reviewed below.

Of course, we are concerned with degenerations of  $K3$  surfaces, thus the KSBA replacement strictly speaking doesn’t make sense (the main issue is the uniqueness of the replacement). Nonetheless, given a degeneration  $\mathcal{X}^*/\Delta^*$  with general fiber a  $K3$ , there exists a filling with  $X'_0$  being a surface with slc singularities (and trivial dualizing sheaf). This follows from the Kulikov-Pinkham theorem and Shepherd-Barron [SB83a, SB83b]. Furthermore, if we start with  $X_0$  a unique non-log canonical singularity, we can apply the same considerations as in Hassett [Has00] and ask for example: *What is the KSBA replacement for a quartic surface  $X_0$  with a single  $E_{12}$  singularity?* In this type of situation, it is easy to see that the resolution  $\hat{X}_0$  is rational (similar to saying that the normalization of a cuspidal cubic curve is rational), and thus the focus is on the “tail” part.



5.5.2. *Triangle or Dolgachev singularities.* The classes of singularities that interests us are instances of the so called Dolgachev singularities [Dol75] (aka triangle singularities or exceptional unimodal singularities). They are essentially the simplest non-log canonical singularities, an analogue of the cuspidal singularity (for curves) in dimension two. The Dolgachev singularity can be defined by their resolution: the (non-minimal) resolution consists of  $E \cup E_1 \cup E_2 \cup E_3$  with  $E^2 = -1$ ,  $E_1^2 = -p$ ,  $E_2^2 = -q$ ,  $E_3^2 = -r$ , where  $(p, q, r)$  are called the Dolgachev numbers of the singularity; the curves  $E_i$  only meet  $E$  transversely (comb type picture). Note that by contracting the  $E_i$ , we obtain a partial resolution with a rational curve  $E$  having 3 quotient singularities of type  $\frac{1}{p}(1, 1)$ ,  $\frac{1}{q}(1, 1)$  and  $\frac{1}{r}(1, 1)$ . There are exactly 14 cases of integers  $(p, q, r)$  leading to hypersurface singularities, these cases are called Dolgachev singularities (or triangle singularities or exceptional unimodal singularities, the latter is the terminology used by Arnold et al. [AGLV98]). The relevant cases here are  $E_{12}$ ,  $E_{13}$ ,  $E_{14}$  corresponding to  $(2, 3, 7)$ ,  $(2, 4, 5)$ , and  $(3, 3, 4)$ .

*Remark 5.19* (The  $T_{p,q,r}$  graphs and lattices). Very relevant in this discussion is the so called  $T_{p,q,r}$  graph (for  $p, q, r$  positive integers). This consists of a central node, together with 3 legs of lengths  $p - 1$ ,  $q - 1$ , and  $r - 1$  respectively. As usual to such a graph, one can associate an even lattice by giving a generator of norm  $-2$  for each node, and two generators are orthogonal unless the corresponding nodes are joined by an edge in the graph (in which case, we define the intersection number to be 1). The cases  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$  corresponding precisely to the ADE Dynkin graphs (with ADE associated lattices). For example  $(1, p, q)$  corresponds to  $A_{p+q-1}$ , while  $(2, 3, 3)$  corresponds to  $E_6$ . Note also  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$  is equivalent to the associated lattice being negative semi-definite. The three cases with  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$  correspond to the extended Dynkin diagrams of type  $\tilde{E}_r$  ( $r = 6, 7, 8$ ), and in these cases the associated lattice is negative semi-definite. Finally, the cases with  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$  lead to a hyperbolic lattice. It is easy to compute that the absolute value of the discriminant will be  $pqr \left(1 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)\right)$ .

The lattice of vanishing cycles associated to a Dolgachev singularity is  $T_{p',q',r'} \oplus U$  for some integers  $(p', q', r')$ , which are called the Gabrielov numbers of the singularity. In particular, we note  $p' + q' + r' = (p' + q' + r' - 2) + 2 = \mu$  the Milnor number of the singularities (i.e. the rank of the lattice of vanishing cycles is the Milnor number). Thus, associated to a Dolgachev singularity there are two triples of integers: the Dolgachev numbers  $(p, q, r)$  related to the resolution of the singularity, and the Gabrielov numbers  $(p', q', r')$  related to the lattice of vanishing cycles (and the local monodromy associated with the singularity). In Table 4 below we give these numbers for the cases relevant to us. Arnold observed that the 14 Dolgachev singularities come in pairs of two with the property that the Dolgachev and Gabrielov numbers are interchanged. This is part the so called strange duality (see [Ebe99] for a survey). The key point is that  $T_{p,q,r}$  and  $T_{p',q',r'}$  are mutually orthogonal in  $E_8^2 \oplus U^2$  (equivalently, after adding a  $U$  to one of them, they can be interpreted as the Neron-Severi lattice and the transcendental lattice for certain  $K3$  surfaces, and thus indeed one has an instance of mirror symmetry for  $K3$  surfaces, see [Dol96]).

Singularity	Dolgachev No.	Gabrielov No.
$E_{12}$	2,3,7	2,3,7
$E_{13}$	2,4,5	2,3,8
$E_{14}$	3,3,4	2,3,9

TABLE 4. The relevant Dolgachev Singularities

Looijenga [Loo83, Loo84] has studied the deformation space for these singularities. Briefly, the triangle singularities are unimodal, i.e. they have 1-parameter equisingular deformation. Within

the equisingular deformation, there is a distinguished point corresponding to the case of  $\mathbb{C}^*$ -action (or equivalently the equation is quasi-homogeneous). Then, one can apply Pinkham's theory of deformations with  $\mathbb{C}^*$ -action. In this situation, there will be 1-dimensional 0-weight direction (i.e. there is an induced  $\mathbb{C}^*$  action on the tangent space to the mini-versal deformation, and the weights refer to this action) corresponding to the equisingular deformations. The remaining  $\mu-1$  weights are negative and correspond to the smoothing directions. We denote by  $S_-$  the germ corresponding to the negative weights. Due to the  $\mathbb{C}^*$ -action  $S_-$  can be globalized and identified to an affine space. Then, clearly  $(S_- \setminus \{0\})/\mathbb{C}^*$  is a weighted projective space of dimension  $\mu - 2$  (where  $\mu$  is the Milnor number, e.g.  $\mu - 2 = 10$  for  $E_{12}$ ). What the general theory of Pinkham says is that  $(S_- \setminus \{0\})/\mathbb{C}^*$  can be interpreted as a moduli space of pairs  $(X, H)$  (where  $H$  is to be interpreted as a hyperplane at infinity coming from a  $\mathbb{C}^*$ -equivariant compactification of the singularity). What Looijenga [Loo83, Loo84] observed is that in the situation of Dolgachev/triangle singularities is that (a general)  $X$  is a  $K3$  surface and  $H$  is a  $T_{p,q,r}$  configuration of rational curves (where as above  $(p, q, r)$  are the Dolgachev numbers). In particular, the transcendental lattice of  $X$  is  $T_{p',q',r'} \oplus U$  (which becomes the lattice of vanishing cycles for the triangle singularity), while  $T_{p,q,r}$  is the Neron–Severi lattice. In conclusion, the weighted projective space  $(S_- \setminus \{0\})/\mathbb{C}^*$  is birational to a locally symmetric variety  $\mathcal{D}/\Gamma$  corresponding to moduli of  $T_{p,q,r}$ -marked  $K3$  surfaces (the dimension is  $20 - (p + q + r - 2) = 22 - (p + q + r) = p' + q' + r' - 2 = \mu - 2$ ). Furthermore, the key point of Looijenga [Loo84] is that the structure of the Baily-Borel compactification  $(\mathcal{D}/\Gamma)^*$  is related to the adjacency of simple-elliptic and cusp singularities to the given Dolgachev/triangle singularity (in line with our discussion from **Section 2**), and that the indeterminacy of the period map  $(S_- \setminus \{0\})/\mathbb{C}^* \dashrightarrow (\mathcal{D}/\Gamma)^*$  is related to the triangle singularities adjacent to the given one (e.g.  $E_{12}$  is adjacent to  $E_{13}$  and this will lead to indeterminacy, that is resolved by Looijenga's theory; while, on the other hand  $E_{12}$  is adjacent only to simple elliptic, cusp, or ADE singularities, and thus there is no indeterminacy).

**Example 5.20.** The simplest case is the deformation of  $E_{12}$ . In this situation, as explained,  $E_{12}$  only deforms to log canonical singularities giving a regular period map, which in turn gives an isomorphism:

$$W\mathbb{P}(w_0, \dots, w_{11}) \cong (S_- \setminus \{0\})/\mathbb{C}^* \cong (\mathcal{D}/\Gamma)^*.$$

The weights  $w_0, \dots, w_{11}$  are the negative weights with respect to the  $\mathbb{C}^*$ -action on the tangent space to the mini-versal deformation of the  $E_{12}$ -singularity, which we recall can be identified with  $\mathcal{O}/J(f)$ , where  $J(f)$  is the Jacobian ideal (here  $f = x^2 + y^3 + z^7$ ). Here  $(\mathcal{D}/\Gamma)^*$  is the Baily-Borel compactification for the moduli space of  $T_{2,3,7}$ -marked  $K3$  surfaces (N.B.  $T_{2,3,7} \cong E_8 \oplus U$ ; also, because of self-duality in this case, the transcendental lattice is  $T_{2,3,7} \oplus U = E_8 \oplus U^2$ ).

We recall that we have defined  $W_8$  (and similarly for  $W_7$  and  $W_6$ ) to be locus of quartics with  $E_{12}$  (and respectively  $E_{13}$  or  $E_{14}$ ) or worse singularities (i.e. the closure of the  $E_{12}$ -equisingular stratum) in the GIT quotient  $\mathfrak{M}$  for quartics. Assuming the universal family of quartics gives a versal deformation for the  $E_{12}$  singularity (this follows from Urabe's analysis [Ura84] of quartics with this type of singularities, or more generally from du Plessis–Wall [dPW00] and Shustin–Tyomkin [ST99]), then at a quartic with  $E_{12}$  singularities such that the singularity has  $\mathbb{C}^*$ -action, the germ of  $(S_-, 0)$  can be interpreted as the normal direction to  $W_8$ . Then,  $(S_- \setminus \{0\})/\mathbb{C}^*$  is nothing but the projectivized normal bundle, which is then the replacement via a (weighted) flip of the  $W_8$  locus. On the other hand, as noted in the example above,  $(S_- \setminus \{0\})/\mathbb{C}^*$  can be interpreted as the moduli of  $T_{2,3,7}$ -marked  $K3$ s, which is the same as our  $Z^9$  locus in  $\mathcal{F}$  (the moduli of quartic  $K3$  surfaces). The same considerations apply to the case of  $E_{13}$  and  $E_{14}$  singularities, but in those cases the identification of  $(S_- \setminus \{0\})/\mathbb{C}^*$  with the moduli of  $T_{2,4,5}$  (and  $T_{3,3,4}$  respectively) marked  $K3$ s (which then correspond to  $Z^7$  and  $Z^6$  respectively) involves one (or respectively two) flips

(corresponding to the fact that  $E_{13}$  deforms to  $E_{12}$ , and similarly for  $E_{14}$ ). This is of course exactly as predicted by [LO16] and the general theory of Looijenga.

The argument above almost establishes our claim that a flip replace the  $Z^9$  locus in  $\mathcal{F}$  (which was identified in [LO16]) by  $W_8$  (the  $E_{12}$  locus) in  $\mathfrak{M}$  (and similarly for  $E_{13}$  and  $E_{14}$ ). In the following subsection, we strengthen the evidence towards this claim by a one-parameter computation (which shows that in deed the generic KSBA replacement for a quartic with  $E_{12}$  singularities is a  $T_{2,3,7}$ -marked  $K3$ ). Furthermore, in the final section of the paper, where we discuss GIT for hyperelliptic quartics, we will give a realization of the flip by means of VGIT, and via Luna's slice theorem  $(S_- \setminus \{0\})/\mathbb{C}^*$  can be interpreted as the replacement of the  $E_k$  loci ( $k = 12, 13, 14$ ) as one moves from GIT towards the Baily-Borel compactification (see [Laz13, §4.2] for a discussion of the local model for VGIT using Luna's theorem).

**5.5.3. The semistable replacement in the  $E_k$  situation ( $k = 12, 13, 14$ ).** We are assuming that we are given a quartic surface  $X_0$  with a unique  $E_k$  singularity (for  $k = 12, 13, 14$ ) and such that the singularity has  $\mathbb{C}^*$ -action (i.e. locally given by the equation in Table 5). We are considering a generic smoothing  $\mathcal{X}/\Delta$  and we are asking what is the KSBA replacement associated to this family. The computation is purely local, similar to that occurring in Hassett [Has00]. We proceed similarly to **Example 5.18**. Namely, a generic smoothing is locally given by  $V(f(x, y, z) + t) \subset (\mathbb{C}^4, 0)$ , where  $f$  is the local equation of the singularity as in Table 5. We make a base change  $t \rightarrow t^k$  so that the local monodromy is unipotent. Arnold et al (see [AGLV98, Table on p. 113]) have computed the spectrum of the singularities for the simplest type of hypersurface singularities, including ours. The spectrum encodes the log of the eigenvalues of the local monodromy, thus from Arnold's list it is immediate to find the base change giving unipotent monodromy; the relevant order  $k$  for the base change is given in Table 5 below. It turns out that the resulting 3-fold  $\mathcal{X} = V(f(x, y, z) + t^k) \subset (\mathbb{C}^4, 0)$  has a simple  $K3$ -singularity (analogue of simple elliptic) at the origin in the sense of Yonemura [Yon90]. Thus a blow-up of  $\mathcal{X}$  (at the origin) will resolve this singularity, giving a  $K3$  tail. In our particular situation (similar to the cuspidal case) it is easy to do this resolution by a weighted blow-up. The tail  $T$  will be one of the weighted  $K3$  surfaces in the sense of M. Reid. What is specific for the situation analyzed here is that  $T$  has 3  $A$ -type singularities lying on a rational curve (the curve corresponding to the exceptional divisor on the weighted blow-up of  $X_0$ ). A routine analysis (see Gallardo [Gal13] for further details) gives:

Singularity	Equation (with $\mathbb{C}^*$ -action)	Order $N$ for base change	Weights $(t, x, y, z)$
$E_{12}$	$x^2 + y^3 + z^7$	42	$(1, 21, 14, 6)$
$E_{13}$	$x^2 + y^3 + yz^5$	30	$(1, 15, 10, 4)$
$E_{14}$	$x^3 + y^2 + yz^4$	24	$(1, 8, 12, 3)$

TABLE 5. The relevant Dolgachev Singularities

**Proposition 5.21.** *Let  $\mathcal{X}/\Delta$  be a generic smoothing of a Dolgachev singularity of type  $E_k$  ( $k = 12, 13, 14$ ). Then, after a base change of order  $k$  as given in Table 5, followed by a weighted blow-up with weights as given in the table, gives a new central fiber  $X'_0$  which is the union of the partial resolution  $\widehat{X}_0$  of  $X_0$  (with quotient singularities as given by the Dolgachev numbers  $(p, q, r)$ ) union a  $K3$  surface  $T$  with 3  $A_*$  singularities:  $A_{p-1}$ ,  $A_{q-1}$ , and  $A_{r-1}$  singularities, which lie on a rational curve  $C = T \cap \widehat{X}_0$ . Thus,  $T$  is a  $T_{p,q,r}$ -marked  $K3$  surface, where  $(p, q, r)$  are the associated Dolgachev numbers with the singularity  $E_k$ .*

*Proof.* The equation of the tail is simply

$$V(f(x, y, z) + t^N) \subset W\mathbb{P}(1, w_x, w_y, w_z)$$

with  $f$ ,  $N$ , and the weights as given in Table 5. Note that this is a weighted degree  $N$  hypersurface in a weighted projective space such that the sum of weights satisfies

$$1 + w_x + w_y + w_z = N.$$

This is precisely the  $K3$  condition.  $\square$

*Remark 5.22.* Computations and arguments of similar nature have been done in the thesis of P. Gallardo (some of them appearing in [Gal13]), who was advised by the first author. We have learned about similar computations done by Shepherd-Barron from an unpublished letter to R. Friedman.

In conclusion, we see that as claimed the replacement of the quartics with  $E_{12}$ ,  $E_{13}$ , and  $E_{14}$  singularities should be  $T_{2,3,7}$ ,  $T_{2,4,5}$ ,  $T_{3,3,4}$  marked  $K3$  surfaces respectively. These coincide precisely with the loci  $Z^9, Z^8, Z^7$  that we have identified in [LO16]. Specifically, we have the following:

**Proposition 5.23.** *The loci  $Z^9, Z^8, Z^7$  are naturally identified with the moduli spaces with the moduli spaces of  $T_{2,3,7}$ ,  $T_{2,4,5}$ ,  $T_{3,3,4}$ -polarized  $K3$  surfaces respectively (in the sense of [Dol96]).*

*Proof.* As already noted, for  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ ,  $T_{p,q,r}$  are hyperbolic lattices of signature  $(1, p + q + r - 3)$ . Furthermore, the absolute value of their discriminant is  $pqr - pq - pr - qr = pqr \left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right)$  (giving values 1, 2, 3 respectively in our situation). It follows, that the three  $T_{p,q,r}$  lattices considered here are isometric to  $E_8 \oplus U$ ,  $E_7 \oplus U$ , and  $E_6 \oplus U$  respectively. Each of them has a unique embedding into the  $K3$  lattice  $E_8^2 \oplus U^3$ , and the corresponding orthogonal complements are  $E_8 \oplus U^2$ ,  $E_8 \oplus U^2 \oplus A_1$ , and  $E_8 \oplus U^2 \oplus A_2$  respectively. This coincides with our definition of the  $Z^9, Z^8, Z^7$  loci from [LO16].  $\square$

## 6. VARIATION OF GIT FOR HYPERELLIPTIC QUARTICS

A hyperelliptic quartic  $S$  is a double cover of a quadric  $Q \subset \mathbb{P}^3$  branched along a divisor in  $|\omega_Q^{-2}|$ , i.e. a  $(2, 4)$ -complete intersection curve. It is clear that  $S$  is determined by (and determines) the curve  $C$ . Thus, we can interpret  $\mathcal{F}_h$ , and more generally  $\mathcal{F}_h(\beta)$ , as a birational model for the moduli space of  $(2, 4)$  complete intersection curves in  $\mathbb{P}^3$ . There are several GIT constructions for moduli spaces of  $(2, 4)$  curves. Namely, an extremal model is that of GIT for  $(4, 4)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$  (which is discussed in Shah [Sha81, Section 4, esp. Thm. 4.8], and partially reviewed in **Subsection 2.3** and **Subsection 3.4** above). At the other extreme there is the GIT for the Chow variety of  $(2, 4)$ -curves in  $\mathbb{P}^3$ . Connecting the two extreme cases, there is a one-parameter VGIT. The situation is similar to that studied in [CMJL14, CMJL12] - where we had a similar setup for  $(2, 3)$  complete intersection curves (the interest in loc. cit. was to study the Hassett–Keel program for genus 4 curves via GIT for canonically embedded genus 4 curves). By identifying these GIT models (for appropriate choices of polarization) with  $\mathcal{F}_h(\beta)$ , we obtain that (a)  $\mathcal{F}_h(\beta)$  are well defined (i.e. the corresponding rings of sections are finitely generated), and (b) that the arithmetic predictions of [LO16] (see **Subsection 1.5**) are actually realized. The results are unconditional in the range  $\beta \in [\frac{1}{4}, 1]$ . In the range  $\beta \in (0, \frac{1}{4}]$  they are conditional on a technical issue, whose analogue for  $(2, 3)$  c.i. curves has been settled in [CMJL14].

**6.1. The GIT set-up and the main results.** Let  $E \rightarrow |\mathcal{O}_{\mathbb{P}^3}(2)|$  be the vector-bundle with fiber  $H^0(\mathcal{O}_Q(4))$  over a quadric  $Q \subset \mathbb{P}^3$ . Thus

$$\mathbb{P}E = \{([f_2], [\bar{f}_4]) \mid f_i \in \Gamma(\mathcal{O}_{\mathbb{P}^3}(i)), \bar{f}_4 = f_4|_{V(f_2)}\}$$

The natural map  $\pi: \mathbb{P}E \rightarrow |\mathcal{O}_{\mathbb{P}^3}(2)| \cong \mathbb{P}^9$  is a  $\mathbb{P}^{24}$ -fibration. The Picard group  $\text{Pic}(\mathbb{P}E)$  is generated by

$$\eta = \pi^* \mathcal{O}(1), \quad \xi = \mathcal{O}_{\mathbb{P}E}(1).$$

For  $t \in \mathbb{Q}$  we let

$$L_t = (\eta + t\xi) \in \text{Pic}(\mathbb{P}E) \otimes_{\mathbb{Z}} \mathbb{Q},$$

and we call  $t$  the slope of  $L_t$ .

**Proposition 6.1** ([Ben14, Thm 2.7]). *The line bundle  $L_t$  on  $\mathbb{P}E$  is ample iff  $t \in (0, \frac{1}{3})$ . Furthermore,  $L_t$  is big and movable if  $t \in (0, \frac{1}{2}]$ .*

*Proof.* The first part is just [Ben14, Theorem 2.7]. The second part follows easily from the computation of the polarization for the Chow variety (which is birational to  $\mathbb{P}E$ ) for  $(2, 4)$  complete intersections in  $\mathbb{P}^3$  (see **Proposition 6.16**).  $\square$

The action of  $\text{PGL}(4)$  on  $\mathbb{P}^3$  defines a  $\text{PGL}(4)$ -action on  $\mathbb{P}E$  and compatible actions on  $\eta$  and  $\xi$ . Thus for each slope for which  $L_t$  is ample we have a GIT quotient

$$(6.1) \quad \mathfrak{M}_h(t) = \mathbb{P}E //_{L_t} \text{SL}(4).$$

For  $t \in [\frac{1}{3}, \frac{1}{2}]$ , one can still define a GIT quotient as the Proj of the ring of invariants, but since the polarization is not ample, special care is needed as discussed in [CMJL14]. In particular, one can define the notion of *numerical  $t$ -stable* (as usual involving the numerical function  $\mu^t([q], [\bar{f}], \lambda)$ ), see [CMJL14, Definition 4.2].

**Definition 6.2.** For  $t_0 \in [0, \frac{1}{2}] \cap \mathbb{Q}$  we say  $t_0$  is a critical value if the numerical  $t$ -stability changes at  $t = t_0$ .

As explained in [Laz13, §4.1], one can see that the set of critical values is finite. For the ample range  $t \in (0, \frac{1}{3})$ , the numerical criterion tells us that the numerical  $t$ -stability is the same as  $L_t$  GIT stability. Furthermore, the critical  $t$  in this range are precisely the  $t$  for which the GIT quotient  $\mathfrak{M}_h(t)$  changes the isomorphism type.

We recall the notation  $\mathcal{F}_h(\beta) = \mathcal{F}(18, \beta) = \text{Proj} R(\mathcal{F}(18), \lambda(18) + \beta\Delta(18))$ . There are natural birational maps

$$(6.2) \quad \mathfrak{p}_h(t): \mathfrak{M}_h(t) \dashrightarrow \mathcal{F}_h \dashrightarrow \mathcal{F}_h(\beta).$$

In fact, the first map is defined for all orbits of smooth curves  $C \in |\mathcal{O}_{\mathbb{P}^1}(4) \boxtimes \mathcal{O}_{\mathbb{P}^1}(4)|$  by considering periods of the double cover of  $\mathbb{P}^1 \times \mathbb{P}^1$  branched over  $C$ , and the second map is obvious. In order to make the connection between  $\mathfrak{M}_h(t)$  and our models  $\mathcal{F}_h(\beta)$ , we define

$$\beta(t) = \frac{1 - 2t}{4t}.$$

**Theorem 6.3.** *Keeping notation as above, the following hold:*

- (1) *Let  $\beta \in [\frac{1}{4}, 1]$ . The ring of sections  $R(\mathcal{F}(18), \lambda(18) + \beta\Delta(18))$  is finitely generated, and the rational map in (6.2) for  $\beta = \beta(t)$  is an isomorphism:*

$$(6.3) \quad \mathcal{F}_h(\beta(t)) \cong \mathfrak{M}_h(t), \quad t \in \left[\frac{1}{6}, \frac{1}{3}\right].$$

*In particular, the variation  $\mathcal{F}_h(\beta)$  for  $\beta \in [\frac{1}{4}, 1]$  is modeled by a variation of GIT with critical  $\beta$  equal to  $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$ , and 1.*

- (2) *For  $\beta = 0$ , the model  $\mathcal{F}_h(0)$  is (by definition) the Baily-Borel compactification. For  $\beta = 1$ , the model  $\mathcal{F}_h(1)$  is isomorphic to the GIT quotient for  $(4, 4)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$ .*
- (3)  *$t \in (0, \frac{1}{2}]$  is a numerical critical value iff  $\beta(t)$  is a critical value according to our predictions in [LO16] (i.e.  $\beta(t) \in \{0, \frac{1}{8}, \frac{1}{6}, \frac{1}{5}, \dots, 1\}$ ).*



*Remark 6.4.* For a countable set of values  $t_d$  (indexed by integer  $d \geq 4$ ; see (6.4)) in the range  $t \in [\frac{1}{3}, \frac{1}{2}]$ , we can prove the finite generation of the ring of sections associated to  $\beta(t)$ , and to identify the corresponding  $\mathcal{F}_h(\beta)$  with the GIT quotient of the Hilbert scheme of  $(2, 4)$  complete intersections in  $\mathbb{P}^3$  for an appropriate choice of polarization. Namely, we recall that the Hilbert scheme is constructed by considering the degree  $d$  piece  $\mathcal{I}_Z(d)$  of the ideal associated to a scheme  $Z$ . For  $d \gg 0$ , this gives an embedding of the Hilbert scheme parameterizing  $Z$  into a Grassmannian, which is canonically polarized by the Plücker embedding. This leads to a countable number of GIT quotients corresponding to the choice of  $d$  (this makes sense for  $d \geq 4$ ). We call this construction the GIT associated to the  $d$ -th Hilbert point. As we discuss in §6.2.3, one can show that these Hilbert GIT differ in codimension 2 from  $\mathcal{F}_h$  and thus they can be identified with some of the  $\mathcal{F}(\beta)$ . For instance (modulo an unproved claim, which we expect to be true) it follows that the GIT quotient  $\text{Chow}_{(2,4)} // \text{SL}(4)$  is isomorphic to the Baily-Borel compactification  $\mathcal{F}_h^*$ . Similarly, the GIT quotient of the Hilbert scheme corresponding to  $d \gg 0$  should recover  $\mathcal{F}_h(\epsilon)$ , the Looijenga's  $\mathbb{Q}$ -factorialization associated to  $\Delta(18)$  (see the discussion of **Section 4**).

*Remark 6.5.* Continuing the previous remark, we note that typically the GIT associated to the  $d$ -th Hilbert point will miss the critical  $\beta$ . The issue is that the slope of the linearization for the  $d$ -th Hilbert point construction depends quadratically in  $d$  (see (6.4) and [CMJL14, Table 1 on p. 732] for a related discussion). To handle this issue, one would have to work with the ample cone of the Hilbert scheme of  $(2, 4)$ -c.i. (and not only the polarizations coming from the  $d$ -th point construction), as well as to allow  $d$  to take small values. Handling these issues for the case of  $(2, 3)$ -c.i. is the main technical content of [CMJL14]. Here we only give a brief discussion in §6.2.3 below. Further work is needed to confirm that the predictions of (3) in the above theorem are indeed complete (the finite generation might be easier to obtain, and we expect to hold in much higher generality). Nonetheless, it is striking that the arithmetic predictions of [LO16] perfectly match the numerical predictions coming from the GIT numerical criterion.

*Remark 6.6.* The parameter space  $\mathbb{P}E$  for  $(2, 4)$ -c.i. is the simplest model of the Hilbert scheme, it corresponds to the 4-th Hilbert point in the language used above. The Hilbert scheme is significantly more involved (see [RV02] for the easier case of  $(2, 3)$ -c.i.). Nonetheless, as in [CMJL14], we expect that numerical GIT on  $\mathbb{P}E$  will suffice for confirming our predictions (N.B. the issue is that outside the ample range, i.e.  $t > \frac{1}{3}$ , one needs additional work to interpret the numerical results).

**6.2. Identification of  $\mathcal{F}(\beta)$  models with GIT models.** In this section, we explain the identification of some  $\mathcal{F}_h(\beta)$  models with GIT models (such as  $\mathfrak{M}_h(t)$  or GIT of Hilbert schemes of  $(2, 4)$ -c.i.). A priori  $\mathcal{F}_h(\beta)$  is not well defined (as we don't know yet finite generation), nonetheless we can prove  $\mathcal{F}_h(\beta)$  is well defined and identify the model by the following recipe: Find a space  $M$  (in our situation it will be a GIT model) such that

- (1) *There is a birational map  $M \dashrightarrow \mathcal{F}_h$  which is an isomorphism in codimension 2.* In practice this means to establish that the double covers of  $\mathbb{P}^1 \times \mathbb{P}^1$  branched over a curve with a single node, or the double covers of the quadric cone branched in a smooth curve are stable. And conversely,  $M$  is parametrizing (possibly away from codimension higher than 1) only  $(2, 4)$  complete intersection curves in  $\mathbb{P}^3$ . In particular, all the models  $\mathfrak{M}_h(t)$  for  $t \in (0, \frac{1}{3}]$  satisfy this second condition.

Once this codimension 2 statement is satisfied, we have an identification

$$N^1(M)_{\mathbb{Q}} \cong N^1(\mathcal{F}_h)_{\mathbb{Q}}$$

(and both will have rank 2, because this is already known for  $\mathcal{F}_h$ ) and thus we can talk of  $L(\beta) = \lambda + \beta\Delta$  as a divisor on  $M$ . With these preliminaries, the needed condition is

- (2)  *$L(\beta)$  is an ample  $\mathbb{Q}$ -Cartier divisor on  $M$ .* In practice,  $L(\beta)$  will be just the polarization on a GIT quotient.

If both conditions are satisfied, then clearly

$$\mathcal{F}_h(\beta) := \text{Proj}(\mathcal{F}_h, L(\beta)) \equiv \text{Proj}(M, L(\beta)) = M,$$

i.e. it is well defined and identified with  $M$ .

In practice, we have a two step process: first identify GIT quotients such that the polarization matches with  $L(\beta)$ . Then check the codimension 2 condition.

*Remark 6.7.* This approach of identifying spaces defined abstractly as Proj of rings of sections with specific GIT quotients is used extensively in the Hassett–Keel program (and arguably, it is the main tool there). Our work in this section follows closely [CMJL14], which is essentially the genus 4 case of the Hassett–Keel program.

In the case of models  $\mathfrak{M}_h(t)$  (with  $t$  in the ample range) we check below that numerically they match with  $\mathcal{F}(\beta)$  for  $\beta = \beta(t)$  defined above. In this situation, it is not hard to check also the codimension 2 condition. Outside, the ample range we can identify further  $\mathcal{F}(\beta)$  with quotients of the Hilbert scheme (and in the limit Chow variety) - this identification is only discrete, but we expect that it can be made continuous as in [CMJL14].

6.2.1. *Divisor Classes on GIT and BB models; Identification of  $\mathcal{F}_h(\beta)$  with  $\mathfrak{M}_h(t)$ .* In [LO16], we have studied the Picard group for all  $\mathcal{F}(N)$  in the  $D$ -tower (see [LO16, Theorem 3.1.1]). In particular, for  $N = 18$ , it follows that the Picard group for  $\mathcal{F}_h = \mathcal{F}(18)$  is generated by the nodal  $H_n$  and the hyperelliptic divisor  $H_h$ . Furthermore, the Borcherds' relation (see [LO16, Theorem 3.1.2]) expresses the Hodge bundle  $\lambda$  in terms of these generators. In summary, the following hold:

**Proposition 6.8** (cf. [LO16, Sect. 3]). *Let  $\mathcal{F}_h$  be the period space of hyperelliptic polarized K3's of degree 4, and set (temporarily)  $H_n := H_n(18)$ ,  $H_h := H_h(18)$ . The following hold:*

- i)  $\text{Pic}(\mathcal{F}_h)_{\mathbb{Q}} = \langle H_n, H_h \rangle_{\mathbb{Q}}$ ;
- ii)  $136\lambda = H_n + 16H_h$ , and (in particular)  $(H_n + 16H_h)$  is an ample  $\mathbb{Q}$ -Cartier divisor on  $\mathcal{F}_h^*$ ;
- iii)  $K_{\mathcal{F}_h} = 18\lambda - \frac{1}{2}(H_n + H_h)$ .

Our prediction is that  $\mathcal{F}_h(\beta)$  is well defined (equivalently, that the ring of sections  $R(\mathcal{F}_h, \lambda + \beta\Delta)$  is finitely generated) and that it differs in codimension 2 from  $\mathcal{F}_h^*$  for  $\beta \in [0, 1)$ . Under these assumptions, for general  $\beta$  (or more precisely for  $\beta$  non-critical),  $\text{Pic}(\mathcal{F}_h(\beta))_{\mathbb{Q}}$  has rank 2 and it is generated by the strict transforms of the nodal and hyperelliptic divisors. By abuse of notation, we will still denote  $\lambda, H_h, H_n$  the classes of these divisors on  $\mathcal{F}_h(\beta)$  (with  $\lambda$  satisfying the same relation as above).

On the other hand it is clear that  $\text{Pic}(\mathbb{P}E)$  has rank 2. For  $t \in (\frac{1}{6}, \frac{1}{3})$  generic (away from critical values)  $\text{Pic}(\mathfrak{M}_h(t))_{\mathbb{Q}}$  has rank 2 being generated by two classes  $\eta$  and  $\xi$ , where  $\eta$  is the descent of the pullback of  $\mathcal{O}(1)$  via  $\mathbb{P}E \rightarrow \mathbb{P}^9$  and  $\xi$  is the relative  $\mathcal{O}(1)$ . The geometric divisors  $H_n$  and  $H_h$  make sense for  $\mathfrak{M}_h(t)$  with  $t \in (\frac{1}{6}, \frac{1}{3})$ . Specifically, they represent the closure of the locus

$$\{(f_2, [f_4]) \in \mathbb{P}E \mid C = V(f_2, f_4) \text{ is singular}\}$$

and

$$\{(f_2, [f_4]) \in \mathbb{P}E \mid Q = V(f_2) \text{ is singular}\}$$

respectively.

**Lemma 6.9.** *In  $\text{Pic}(\mathfrak{M}_h(t))_{\mathbb{Q}}$ , the following relations hold:*

$$\begin{aligned} \mathfrak{p}_h(t)^* H_h &= 4\eta \\ \mathfrak{p}_h(t)^* H_n &= 72\eta + 68\xi. \end{aligned}$$

*Proof.* A similar computation (for  $(2, 3)$  complete intersections in  $\mathbb{P}^3$  was done in [CMJL12, Prop. 1.1], we omit the details.  $\square$

We now can justify the relationship between the two variations of models:

**Corollary 6.10.** *In  $\text{Pic}(\mathfrak{M}_h(t))_{\mathbb{Q}}$ ,*

$$\eta + t\xi = 2t\mathfrak{p}_h(t)^*(\lambda + \beta(t)\Delta_h),$$

where  $\beta(t) = \frac{1-2t}{4t}$ .

*Proof.* We recall,  $\Delta_h = \frac{1}{2}H_h$ , and thus  $\mathfrak{p}_h(t)^*\Delta_h = 2\eta$ . On the other hand,

$$\mathfrak{p}_h(t)^*\lambda = \frac{1}{136}\mathfrak{p}_h(t)^*(H_n + 16H_h) = \frac{1}{136}(72\eta + 68\xi + 64\eta) = \eta + \frac{1}{2}\xi.$$

Thus,

$$\mathfrak{p}_h(t)^*(\lambda + \beta(t)\Delta_h) = \eta + \frac{1}{2}\xi + \left(\frac{1}{2t} - 1\right)\eta = \frac{1}{2t}(\eta + t\xi)$$

□

**Corollary 6.11.** *For  $t \in (\frac{1}{6}, \frac{1}{3})$ , the ring of sections  $R(\mathcal{F}_h, \lambda + \beta(t)\Delta_h)$  is finitely generated and the following holds*

$$\mathcal{F}_h(\beta(t)) \cong \mathfrak{M}_h(t)$$

*Proof.* We apply the general strategy outlined above to the open subset  $U \subset \mathcal{F}_h$  parameterizing quartic  $K3$  with either at worst a single OPD, or smooth hyperelliptic. Clearly  $U$  is an open subset of  $\mathcal{F}_h$  and  $\text{codim}_{\mathcal{F}_h}(\mathcal{F}_h \setminus U) = 2$ . It follows that  $R(\mathcal{F}_h, \lambda + \beta\Delta_h) = R(U, (\lambda + \beta\Delta_h)|_U)$ . On the other hand, for  $t \in (\frac{1}{6}, \frac{1}{3})$  the curves in  $\mathbb{P}^1 \times \mathbb{P}^1$  with a single node, or the smooth curves on the quadric cone are stable. Thus,  $U$  can be identified with an open subset of  $\mathfrak{M}_h(t)$  for  $t \in (\frac{1}{6}, \frac{1}{3})$ . Clearly,  $U$  has complement of codimension larger than 2 in  $\mathfrak{M}_h(t)$  (this is clear in  $\mathbb{P}E$ ). Now,  $L(t) = \eta + t\xi$  is ample on  $\mathfrak{M}_h(t)$ , and the claim follows. □

**6.2.2. The extremal case corresponding to the GIT quotient for  $(4, 4)$ -curves on  $\mathbb{P}^1 \times \mathbb{P}^1$ .** The parameter  $t$  involved in the variation of GIT quotients  $\mathfrak{M}_h(t)$  has a simple geometric interpretation:  $\frac{1}{t}$  is the weight given to the quadric  $Q$  containing a  $(2, 4)$ -c.i.  $C$ . Since a quadric is GIT semistable iff it is smooth, it is clear that for small  $t$ , the GIT quotient  $\mathfrak{M}_h(t)$  can be identified with the GIT quotient for  $(4, 4)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$  (analyzed by Shah [Sha81, Section 4], and partially reviewed above). The completely analogous situation for  $(2, 3)$ -complete intersection curves was analyzed in [Fed12] and [CMJL14, Subsection 3.2]. We obtain:

**Proposition 6.12.** *For  $t \in (0, \frac{1}{6}]$ ,  $\mathfrak{M}_h(t)$  is naturally isomorphic to the GIT quotient of  $(4, 4)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$ .*

*Proof.* The statement of the proposition amounts to saying that a curve  $(2, 4)$  is  $t$ -semistable in the given range iff it is contained in a smooth quadric, and it is semistable as  $(4, 4)$ -curve (as classified by Shah [Sha80, Thm. 4.8]). The numerical criterion discussed in **Subsection 6.3** below establishes that  $t = \frac{1}{6}$  is the first critical slope. Thus,  $\mathfrak{M}_h(t)$  for  $t \in (0, \frac{1}{6})$  are all isomorphic (no change of stability occurs in the VGIT in this range). It is immediate to see that a curve sitting on a singular quadric is not  $t$ -semistable for  $t < \frac{1}{6}$ . Conversely, it is easy to see (via the numerical criterion) that a curve with a single node on a smooth quadric is  $t$ -stable for all  $t > 0$ . This establishes that  $\mathfrak{M}_h(t)$  and  $\mathfrak{M}_h$  (the GIT quotient for  $(4, 4)$  curves on  $\mathbb{P}^1 \times \mathbb{P}^1$ ) are isomorphic in codimension 1. They are isomorphic over  $U$  the locus of curves with a single node sitting on the smooth quadric. Since  $U$  has Picard number one, the polarizations on  $\mathfrak{M}_h(t)$  and  $\mathfrak{M}_h$  agree on  $U$ , and we conclude as above  $\mathfrak{M}_h(t) \cong \mathfrak{M}_h$ . □

6.2.3. *Identification of  $\mathcal{F}_h(\beta)$  with quotients of Hilbert schemes.* As discussed in **Remark 6.4**, GIT quotients of the Hilbert scheme (and in the limit Chow variety) can be used to identify  $\mathcal{F}_h(\beta)$  for a countable number of values of  $\beta$ . As discussed (in analogy to [CMJL14]), this is the natural approach to take in order to confirm the [LO16] predictions for hyperelliptic quartics. We plan to investigate this in future work. For now, we only discuss the basic set-up.

**Notation 6.13.** We denote by  $\text{Hilb}_{(2,4)}$  (the main component of) the Hilbert scheme for  $(2,4)$ -complete intersections in  $\mathbb{P}^3$ . For  $d \in \mathbb{Z}_+$ , and  $C$  a generic  $(2,4)$  curve, we let  $k_d = \dim H^0(\mathcal{I}_C(d))$ , and  $N_d = \dim H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ . For  $d \geq 4$ , we denote by  $\text{Hilb}_{(2,4)}^d$  the closure of the image of the natural map

$$\text{Hilb}_{(2,4)} \dashrightarrow \text{Gr}(k_d, N_d)$$

sending a scheme  $Z$  to the  $d$ -graded part of the associated ideal. Similarly, we denote by  $\text{Chow}_{(2,4)}$  the associated Chow variety.

We note that (for  $d \geq 4$ )  $\text{Hilb}_{(2,4)}^d$  is a birational model for  $\text{Hilb}_{(2,4)}$ . In fact, it is isomorphic to  $\mathbb{P}E$  for  $d = 4$ , and to  $\text{Hilb}_{(2,4)}$  for  $d \gg 0$ .  $\text{Hilb}_{(2,4)}^d$  is naturally polarized by the restriction of the Plücker line bundle on  $\text{Gr}(k_d, N_d)$ . We refer to the GIT quotient (with respect to this polarization)

$$\text{Hilb}_{(2,4)}^d // \text{SL}(4)$$

as the GIT quotient of the  $d$ -th Hilbert point. Similarly, by construction  $\text{Chow}_{(2,4)}$  is naturally polarized, and thus the GIT quotient  $\text{Chow}_{(2,4)} // \text{SL}(4)$  has an unambiguous meaning. Note that by taking  $d \rightarrow \infty$ , we obtain the natural Hilbert-to-Chow morphism. Combined with basic VGIT, we obtain:

**Lemma 6.14.** *There exists  $d_0 \gg 0$  such that  $\text{Hilb}_{(2,4)}^d // \text{SL}(4)$  are all isomorphic for  $d \geq d_0$ . Furthermore, for  $d \geq d_0$ , the Hilbert to Chow morphism induces a regular morphism  $\text{Hilb}_{(2,4)}^d // \text{SL}(4) \rightarrow \text{Chow}_{(2,4)} // \text{SL}(4)$*

*Proof.* For  $d \gg 0$ ,  $\text{Hilb}_{(2,4)}^d // \text{SL}(4)$  is obtained by varying the linearization on the Hilbert scheme  $\text{Hilb}_{(2,4)}$ . The Chow GIT quotient  $\text{Chow}_{(2,4)} // \text{SL}(4)$  can also be interpreted as a GIT quotient of  $\text{Hilb}_{(2,4)}$  but with respect to a semi-ample line bundle (the one coming via pull-back from the Hilbert-to-Chow morphism). VGIT gives a rational polyhedral decomposition of the ample cone on  $\text{Hilb}_{(2,4)}$ . In fact, the decomposition into rational polyhedral chambers can be extended to the entire Neron-Severi group (see the discussion of [Laz13, §4.1]). The claim now follows from the fact that the polarizations on  $\text{Hilb}_{(2,4)}^d$  converge to the Chow polarization as  $d \rightarrow \infty$  (and the variation of polarizations is 1-parameter).  $\square$

The following is an unproved claim, that we expect to hold by exactly the same arguments as [CMJL14, Prop. 5.2] (with only minor adjustments).

**Claim 6.15.** *For any  $d \geq 4$ , if a scheme  $Z$  is semistable for the GIT quotient  $\text{Hilb}_{(2,4)}^d // \text{SL}(4)$  then it is a  $(2,4)$ -complete intersection. The same holds for the Chow quotient  $\text{Chow}_{(2,4)} // \text{SL}(4)$ .*

Assuming the claim, we can identify various  $\mathcal{F}_h(\beta)$  with GIT quotients of models of the Hilbert scheme (or Chow variety). For instance, using the standard computations of Mumford–Knudsen [KM76], it is possible to determine the values  $\beta_d = \beta(t_d)$  (for  $d \geq 4$ ) such that  $\mathcal{F}_h(\beta_d) \cong \text{Hilb}_{(2,4)}^d // \text{SL}(4)$  (see (6.4) below). Somewhat surprisingly we get that the extremal value corresponding to the Baily-Borel compactification  $\mathcal{F}_h^*$  is precisely the Chow quotient (i.e.  $t_\infty = \frac{1}{2}$  or equivalently  $\beta_\infty = \beta(\frac{1}{2}) = 0$ ). Namely,

**Proposition 6.16.** *The Chow line bundle has slope  $t = \frac{1}{2}$ .*

*Proof.* The computation is standard and completely analogous to [CMJL12, Thm. 2.11].  $\square$

*Remark 6.17.* In fact, the same computation as in [CMJL14] (just using  $(2, 4)$  instead of  $(2, 3)$  as degrees for the complete intersection) give that for  $d$  large enough the slope of  $\text{Hilb}_{(2,4)}^d // \text{SL}(4)$  is

$$(6.4) \quad t_d = \frac{(d-3)^2}{2(d^2 - 4d + 5)}$$

As  $d \rightarrow \infty$ , we recover the slope  $t_\infty = \frac{1}{2}$  corresponding to the Chow polarization.

Thus, we obtain a GIT description for the Baily-Borel compactification  $\mathcal{F}_h^*$ .

**Corollary 6.18.** *Assuming 6.15, The GIT quotient  $\text{Chow}_{(2,4)} // \text{SL}(4)$  of the Chow variety for  $(2, 4)$ -c.i. in  $\mathbb{P}^3$  is isomorphic to the Baily-Borel compactification  $\mathcal{F}_h^*$ .*

**6.3. Numerical Criterion and Critical GIT slopes.** The purpose of this section is to determine the critical slope for the VGIT analysis of  $\mathfrak{M}_h(t)$ . For  $t \in (0, \frac{1}{3}]$  this is a standard VGIT analysis (the only thing that needs some explaining is the explicit form of the numerical criterion for our situation - which is due to Benoist [Ben14]). For  $t \in (\frac{1}{3}, \frac{1}{2}]$ , the numerical computation can be extended, but a priori it doesn't have a GIT meaning (since the corresponding line bundle is not ample). In [CMJL14], we have shown that this virtual critical slopes are in fact realized (the main tool is to compare with GIT quotients of Hilbert schemes). We expect that this analysis can be done also in our situation. In absence of this, the critical values between  $t \in (\frac{1}{3}, \frac{1}{2}]$  should be understood as GIT predictions - in the spirit of [AFS10] (while the methods there are different, the assumptions and predictions in our situation and theirs have the same nature).

The main result of the section is

**Theorem 6.19.** *The critical values for the VGIT  $\mathfrak{M}_h(t)$  are*

$$\left\{ \frac{1}{6}, \frac{1}{4}, \frac{3}{10}, \frac{1}{3}, \frac{5}{14}, \frac{3}{8}, \frac{2}{5}, \frac{1}{2} \right\}$$

*The values  $\leq \frac{1}{3}$  are realized as critical values in a genuine VGIT, while those  $> \frac{1}{3}$  are virtual (N.B. we fully expect that they are realized by an extended VGIT as in [CMJL14]).*

Explicitly, via the correspondence between  $\beta$  and  $t$  discussed in the previous section, we obtain

$t$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{1}{3}$	$\frac{5}{14}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{1}{2}$
$\beta = \frac{1-2t}{4t}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{8}$	0

We conclude with the following corollary, which we view as a very strong evidence for [LO16] predictions for the case of hyperelliptic quartics.

**Corollary 6.20.** *The arithmetic predictions of [LO16] on the critical values for the variation of  $\mathcal{F}(\beta)$  match the predictions for the VGIT of  $\mathfrak{M}_h(t)$  (via  $\beta = \beta(t)$ ).*

*Remark 6.21.* For the range  $\beta \in [\frac{1}{4}, 1]$ , we prove that the predictions in **Corollary 6.20** do hold.

**6.3.1. Numerical Criterion for  $(2, 4)$ -complete intersections.** The main tool to study the stability conditions for GIT problems is the numerical function of Mumford, which in our situation will take the shape  $\mu^t(q, [f_4]; \lambda)$ , where  $t \in (0, \frac{1}{3})$  is the slope of the polarizing line bundle  $L_t$ ,  $(q, f_4)$  are the defining equations for the  $(2, 4)$ -complete intersection, and  $\lambda$  denotes a 1-PS of  $\text{SL}(4)$ . Note that we use the notation  $[f_4]$  to emphasize the fact that  $[f_4]$  is an equivalence class, namely the equation  $f_4$  is taken modulo  $q$  ( $f_4 \sim f'_4$  iff  $f'_4 = f_4 + qq'$  for some  $q' \in H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ ).



As usually, a curve  $C = V(q, f_4)$  is  $t$ -semistable iff  $\mu^t(q, [f_4]; \lambda) \geq 0$  for all  $\lambda$  1-PS. However, due to the choice of representative for  $[f_4]$  one needs to take some care. Specifically,

**Lemma 6.22** (Benoist [Ben14, Prop. 2.15]). *With notations as above, the Mumford numerical function can be computed by*

$$\mu^t(q, [f_4]; \lambda) = \mu(q, \lambda) + t\mu(f_4, \lambda)$$

where  $f_4$  is a representative of  $[f_4]$  of minimal  $\lambda$ -weight, and  $\mu(q, \lambda)$  and  $\mu(f_4, \lambda)$  have the usual meaning (see [MFK94]).

*Remark 6.23.* We will write  $\mu^t(q, f_4; \lambda) (= \mu(q, \lambda) + t\mu(f_4, \lambda))$  for the situation when a representative  $f_4$  was chosen. Obviously,  $\mu^t(q, [f_4]; \lambda) \geq \mu^t(q, f_4; \lambda)$  for any choice of representative  $f_4$ . In practice, we might work as well with  $\mu^t(q, f_4; \lambda)$ . This is so because, as always, to establish semistability/stability one argues by contraposition. Thus, given a curve  $C$  one tries to destabilize it, which involves the choice of special equations  $q$  and  $f_4$  for  $C$ , and then of an adapted 1-PS  $\lambda$ . If  $\mu^t(q, f_4; \lambda) < 0$  then  $C$  is unstable.

As already discussed, for  $t \in (0, \frac{1}{3}]$  we are in a well posed VGIT set-up. We are however interested in the range  $t \in (0, \frac{1}{2}]$  for which we can still define a GIT quotient  $\mathfrak{M}_h(t)$ , but a priori it is not clear if it well defined, or if it has the required properties. Similarly, we can easily extend the numerical criterion to the range  $t \in (0, \frac{1}{2}]$ , but we strongly caution the reader that these are a priori only numerical predictions (of course for  $t \in (0, \frac{1}{3}]$ , they have the usual meaning).

**Definition 6.24.** Let  $C$  be a  $(2, 4)$ -complete intersection. For  $t \in (0, \frac{1}{2}]$ , we say  $C$  is *numerically  $t$ -semistable* (respectively *numerically  $t$ -stable*) if for any choice of equations  $q, f_4$  for  $C$  and any  $\lambda$  1-PS, it holds  $\mu^t(q, [f_4]; \lambda) \geq 0$  ( $> 0$  respectively).

*Remark 6.25.* Similarly to [CMJL14], we fully expect that the numerical (semi)stability is actually GIT (semi)stability, but this needs a proof. Our expectation is based on the fact that the base locus of  $L_t$  (for  $t > \frac{1}{3}$ ) corresponds to highly singular curves (or not even curves –  $q$  and  $f_4$  share a linear factor), which should not be visible in the GIT quotients  $\mathfrak{M}_h(t)$ .

**6.3.2. Basic lemmas for  $t$ -semistability.** In this paper, we are not attempting a full investigation of the GIT stability for the VGIT on  $\mathbb{P}E$ . The focus here is on detecting the critical values  $t$  for which the GIT stability changes. We have a number of easy preliminary lemmas (mostly direct adaptations of similar results from [CMJL14, Section 4]) which establish a basic geometric framework.

First, we note that curves sitting on a smooth quadric do not produce minimal orbits relevant to the change of stability in the interval  $(0, \frac{1}{2}]$ .

**Lemma 6.26.** *Let  $C$  be a curve lying on a smooth quadric. Then  $C$  is numerically  $t$ -polystable for  $t \in (0, \frac{1}{2}] \cap (0, t_C)$  for some  $t_C \geq 0$  (N.B. this includes the case of empty set (if  $t_C = 0$ ) or full interval  $(0, \frac{1}{2}]$ ).*

*Proof.* By the general properties of the numerical function (e.g. see the discussion [Laz13, §4.1]), it is immediate to see that  $C$  will be numerically semistable on an interval of type  $(0, \frac{1}{2}] \cap [t_1, t_2]$ . The content of the lemma is to say that  $t_1 = 0$  (i.e. as  $t$  increases  $C$  can not change from unstable to semistable), and that at  $t_2$  the orbit of  $C$  can not be closed. Specifically,  $t_1$  and  $t_2$  are determined by the conditions: there exists choice of equations  $q$  and  $f_4$  for  $C$  and a choice a 1-PS  $\lambda$  such that the linear function (in  $t$ )  $\mu^t(q, f_4; \lambda)$  changes sign at  $t_i$  ( $t_1$  corresponds to moving from negative to positive, and  $t_2$  from positive to negative). (Note,  $t_1$  and  $t_2$  are obtained by considering all choices of equations and  $\lambda$ , but it is not hard (see [Laz13, §4.1]) that combinatorially only finitely choices are involved and thus indeed one can choose equations and 1-PS  $\lambda$  as above).

Since a smooth quadric is semistable (for the GIT on quadric surfaces), we get  $\mu(q, \lambda) \geq 0$ . Thus,  $\mu^t(q, f_4; \lambda)$  can never change sign from negative to positive, and we can formally set  $t_1 = 0$ . We set  $t_C = t_2 \in [0, \infty]$ . The new polystable orbits occurring at  $t_C$  will be semistable on an interval of type  $[t_C, t_2]$  (typically  $t_2 = t_C$ , and thus semistable only at  $t_C$ ). As discussed above, if a curve  $C_0$  sitting on a smooth quadric is semistable for some  $t_0 > 0$ , it is semistable on the entire interval  $(0, t_0]$ . We conclude that the new polystable curves occurring at  $t_C$  can not be curves on the smooth quadric surface, completing the proof.  $\square$

Next, we exclude the reducible quadrics (see [CMJL14, Prop. 4.6]):

**Lemma 6.27.** *If  $q$  is a reducible quadric, then  $(q, f_4)$  is not numerically  $t$ -semi-stable for any  $t < \frac{1}{2}$ . Moreover, if  $q$  and  $f_4$  share a linear factor, then  $(q, f_4)$  is unstable for all  $t \leq \frac{1}{2}$ .*

*Proof.* Suppose that  $q$  is singular along the line  $V(x_2, x_3)$ . Consider the 1-PS  $\lambda = (1, 1, -1, -1)$ . We have

$$\mu(q, \lambda) = -2, \quad \mu(f_4, \lambda) \leq 4$$

The first statement follows. Similarly, if  $q$  and  $f_4$  share the plane  $V(x_3)$ , we get  $\mu(f_4, \lambda) \leq 2$ , and the claim follows.  $\square$

Similarly, we can exclude the case  $q$  and  $f$  simultaneously singular (cf. [CMJL14, Prop. 4.7])

**Lemma 6.28.** *If  $q$  and  $f_4$  are simultaneously singular, then  $(q, f_4)$  is not numerically  $t$ -semi-stable for any  $t < \frac{1}{2}$ .*

*Proof.* Suppose that the common singularity is at  $(1, 0, 0, 0)$ . Consider the 1-PS  $\lambda = (3, -1, -1, -1)$ . It follows

$$\mu(q, \lambda) \leq -2, \quad \mu(f_4, \lambda) \leq 4$$

Thus  $-2 + t \cdot 4 < 0$  if  $t < \frac{1}{2}$ .  $\square$

**6.3.3. Identification of critical slopes  $t \in (0, \frac{1}{2})$ .** We are interested in describing the VGIT in the interval  $(0, \frac{1}{2}]$ . If  $t \in (0, \frac{1}{2})$  is a critical there exist a minimal orbit  $C = V(q, f_4)$  and a 1-PS  $\lambda$  stabilizing it. Thus

$$\mu^t(q, f_4) = 0$$

and then

$$(6.5) \quad t = -\frac{\mu(q, \lambda)}{\mu(f_4, \lambda)}.$$

It is clear that we can choose equations for  $C$  so  $q$  and  $f_4$  are weighted homogeneous with respect to  $\lambda$  (of weights  $\mu(q, \lambda)$  and  $\mu(f_4, \lambda)$  respectively). Furthermore, since according to the lemmas of the previous subsection any relevant minimal orbits involve only curves sitting on the quadric cone, we can assume  $q = x_0x_2 - x_1^2$  (the standard quadric cone).

As usual we can diagonalize  $\lambda$  and let  $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  be the corresponding exponents (i.e.  $\lambda(t) = \text{diag}(\exp(t\alpha_i))_i \in \text{SL}(4)$ ). Without loss of generality, we can assume the following:

- $\alpha_3 \leq \alpha_2 \leq \alpha_0$ ;
- $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0$ ;
- $2\alpha_1 = \alpha_0 + \alpha_2$ .

The first two conditions are standard, while the last simply says that  $q$  is weighted homogeneous with respect to  $\lambda$ . Since for the numerical criterion we can consider  $(\alpha_i)_i$  up to rescaling, it is immediate to see that we can give a simple 1-parameter parameterization for  $\alpha_i$ , namely:

$$(6.6) \quad \alpha_0 = \alpha, \quad \alpha_1 = 1 \quad \alpha_2 = 2 - \alpha \quad \alpha_3 = -3$$

for  $\alpha \in [1, 5]$ . The two extremal 1-PS's are:  $(1, 1, 1, -3)$  and  $(5, 1, -3, -3)$ .

*Remark 6.29.* Recall that in order to interpret geometrically the meaning of the numerical criterion, one considers the flag associated to the 1-PS  $\lambda$  (see [MFK94, §2.2]). The numerical criterion will essentially measure the incidence of the curve  $C$  with this linear flag. For instance,  $\lambda = (1, 1, 1, -3)$  (and its dual  $\lambda^{-1} = (-1, -1, -1, 3)$ ) will measure how singular the curve is along the hyperplane  $x_3 = 0$  (and respectively at the point  $(0, 0, 0, 1)$ ). In this case, the thing that will be measured is the multiplicity.

The only choice now is  $f_4$  which has to be weighted homogeneous with respect to  $\lambda$ . Assuming that the support of  $f_4$  contains 2 monomials  $x_0^{i_0} \dots x_3^{i_3}$  and  $x_0^{j_0} \dots x_3^{j_3}$ , the weighted homogeneous condition leads to the equation

$$(6.7) \quad \alpha(k_0 - k_2) = 3k_3 - 2k_2 - k_1$$

for  $\alpha$ , where  $k_l = i_l - j_l$  for  $l = 0, \dots, 3$ . This determines  $\alpha$ , and then using (6.6) the 1-PS  $\lambda$ , and finally  $t$  (using (6.5)). In the situation that  $f_4$  is supported on a single monomial, then the condition of weighted homogeneous is void, but then we have to consider the extremal values for  $\alpha$ , namely  $\alpha = 1$  or  $\alpha = 5$ .

This gives a simple algorithm for determining the possible list of critical  $t$ :

**Algorithm 6.30.** *The following procedure determines a set of values that contains all critical  $t \in (0, \frac{1}{2})$ .*

- (1) *List all pairs of distinct degree 4 monomials  $(i_0, i_1, i_2)$  and  $(j_0, j_1, j_2)$ . Solve for  $\alpha$  in equation (6.6). Keep the pairs for which  $\alpha \in [1, 5]$ .*
- (1') *Consider also  $\alpha = 1$  and any monomial  $(i_0, i_1, i_2)$ , and  $\alpha = 5$  and any monomial  $(i_0, i_1, i_2)$ .*
- (2) *For  $\alpha$  and associated monomials as above, compute  $t$  by using (6.6) and (6.5). Keep the values of  $t$  in the range  $(0, \frac{1}{2})$ .*
- (3) *Remove duplicates, and return (the ordered) list of  $t \in (0, \frac{1}{2})$ .*
- (4) *For each  $t$  in the list, return also the relevant 1-parameter subgroups  $\lambda$ , and for each  $\lambda$  the list of weighted degree 4 monomials of weight  $-\frac{\mu(q, \lambda)}{t}$ .*

*Proof.* The function  $\mu^t(q, f_4; \lambda)$  is linear in  $t$  and piecewise linear in  $\lambda$  (if we parametrize  $\lambda$  as in (6.6)). It is clear that at a critical  $t$ , the relevant  $\lambda$  is a critical value for the piecewise linear numerical function, i.e. either it solves the equation (6.7) (this is where the PL function changes rule) or it is corresponding to the boundary values  $\alpha = 1, 5$ . The above algorithm determines the possible critical  $\lambda$ , and then the associated critical  $t$ .  $\square$

**Proposition 6.31.** *The list of  $t \in (0, \frac{1}{2})$  for which the numerical stability changes is included in the set*

$$\left\{ \frac{1}{6}, \frac{1}{4}, \frac{3}{10}, \frac{1}{3}, \frac{5}{14}, \frac{3}{8}, \frac{2}{5} \right\}.$$

*Proof.* Computer implementation of the above algorithm.  $\square$

*Remark 6.32.* Of course, a priori the above list might contain values  $t$  where the stability condition does not change, and thus should be removed. In this particular situation, this is not the case. For this, one needs a second computer program to check the stability at each of these (possible) critical values (e.g. one needs to assure that a generic equation with a given monomial support is not destabilized by some other 1-PS  $\lambda'$ ), followed by a geometric analysis to interpret intrinsically the combinatorial possibilities found by computer.

**6.3.4. List of possible minimal orbits and their geometry.** As explained in the above remark, by applying another standard GIT algorithm (similar to [Laz09a]), we can identify the minimal orbits relevant for each of the  $t$  identified as potential critical slopes in **Proposition 6.31**. For all, except one case, there is only one such orbit. We discuss here only one example:

**Example 6.33** (Case  $E_{12}$ ). Consider the curve

$$C = V(x_0x_2 - x_1^2, x_0x_3^3 - x_1x_2^3).$$

Solving equation (6.7), we obtain that for  $\alpha = 4 \in [1, 5]$  (corresponding to the 1-PS  $\lambda = (4, 1, -2, -3)$ ), the two equations are weighted homogeneous. We have  $\mu(q, \lambda) = 2$  and  $\mu(f_4, \lambda) = -5$  giving the critical value  $t = \frac{2}{5}$ . It is not hard to check that indeed this equation is numerically  $t$ -semistable exactly for  $t = \frac{2}{5}$ . Let's now see the geometric properties of  $C$ . First, consider the vertex of the quadric cone  $v = [0 : 0 : 0 : 1]$ . Considering affine coordinates (i.e. set  $x_3 = 1$ ), we get the equations

$$\begin{aligned} 0 &= x_1^2 - x_0x_2 \\ x_0 &= -x_1x_2^3 \end{aligned}$$

which then give the equation of  $C$  at  $v$ :

$$x_1^2 + x_1x_2^4 = 0$$

which is precisely the equation of a curve with an  $A_7$  singularity. Furthermore, note that  $C$  contains a line which is tangent (with maximal multiplicity) to the residual curve. We now recall that the  $K3$  surfaces obtained as double covers of the quadric cone branched over a curve with these geometric properties ( $A_7$  singularity at the vertex and containing a line) are precisely the  $K3$  surfaces parametrized by  $Z_h^8$  (see **Proposition B.1(2)**). The curve  $C$  considered here does not have ADE singularities (and thus does not lead to a  $K3$ ). In fact, it has an  $E_{12}$  singularity at  $p = [1 : 0 : 0 : 0]$ . Specifically, in affine coordinates at  $p$  (i.e.  $x_0 = 1$ ), we get

$$\begin{aligned} x_2 &= x_1^2 \\ 0 &= x_2^3 + x_1x_2^3 \end{aligned}$$

which gives

$$x_2^3 + x_1^7 = 0,$$

which indeed is an  $E_{12}$  singularity. In conclusion, we see that VGIT at the critical value  $t = \frac{2}{5}$  replaces the locus of curves with  $E_{12}$  singularities by  $K3$  surfaces in the locus  $Z_h^8$ . In other words, assuming that the numerical stability is indeed GIT stability (N.B.  $\frac{2}{5} > \frac{1}{3}$ ), we see that  $W_{h,7}$  is replaced by  $Z_h^8$  as predicted.

The other cases are similar. Due to the choice of normalization for the 1-PS  $\lambda$ , always the relevant points on  $C$  are the vertex  $v = [0 : 0 : 0 : 1]$  of the quadric cone  $V(x_0x_2 - x_1^2)$  and  $p = [1 : 0 : 0 : 0]$ . We conclude:

**Theorem 6.34.** *With notations as above, the values of  $t$  for which the numerical stability changes are  $\{\frac{1}{6}, \frac{1}{4}, \frac{3}{10}, \frac{1}{3}, \frac{5}{14}, \frac{3}{8}, \frac{2}{5}\}$ . Additionally, the numerically polystable orbits that exist only for the given  $t$  are given in Table 6. The associated curve  $C$  has singularities at  $p$  and  $v$  as given in the table.*

*Remark 6.35.* As in the  $E_{12}$  example, we note that the conditions on the singularities at  $v$  are precisely the conditions defining the  $Z_h^k$  loci (see **Proposition B.1**). For instance, the case  $t = \frac{1}{6}$  says that the curve  $C$  lies on a quadric cone, but the vertex  $v$  does not belong to  $C$ , which is precisely the definition of  $Z_h^1 \setminus Z_h^2$ , as predicted. Similarly, the condition on the singularity at  $p$  is precisely the definition of the  $W_{h,k}$  loci (see **Proposition 3.12**).

*Remark 6.36.* In order to fully confirm the predictions of [LO16] in the hyperelliptic case by means of VGIT, two things are needed. First, one needs to verify that the numerical stability is indeed GIT stability for  $t \in (\frac{1}{3}, \frac{1}{2}]$ . As in [CMJL14], one natural approach would be to relate to stability on the Hilbert scheme of  $(2, 4)$  complete intersections as sketched above. Then, one would like to show that the properties given in Table 6 characterize the curves that change stability at a given  $t$ . For instance, one would like to show that a curve with  $E_{12}$  singularity on a quadric (which does

$t$	$\beta$	Sing. at $p$	Sing. at $v$	Min. Orbit
$\frac{1}{6}$	1	quadruple conic	$v \notin C$	$V(x_3^4, x_1^2 - x_0x_2)$
$\frac{1}{4}$	$\frac{1}{2}$	triple conic	$A_1$	$V(x_3^3x_1, x_1^2 - x_0x_2)$
$\frac{3}{10}$	$\frac{1}{3}$	$E_{4,\infty}$	$A_2$	$V(x_0x_3^3 + x_2^2x_3^2, x_1^2 - x_0x_2)$
$\frac{1}{3}$ (generic)	$\frac{1}{4}$	$E_{3,0}$	$A_3$	$V(x_0x_3^3 + 2\alpha x_1x_2x_3^2 + \beta x_2^3x_3, x_1^2 - x_0x_2)$
$\frac{1}{3}$ (special)	$\frac{1}{4}$	$E_{3,\infty}$	double twisted cubic	$V(x_0x_3^3 + 2x_1x_2x_3^2 + x_2^3x_3, x_1^2 - x_0x_2)$
$\frac{5}{14}$	$\frac{1}{5}$	$E_{14}$	$A_4$	$V(x_2^4 + x_0x_3^3, x_1^2 - x_0x_2)$
$\frac{3}{8}$	$\frac{1}{6}$	$E_{13}$	$A_5$	$V(x_1x_2^2x_3 + x_0x_3^3, x_1^2 - x_0x_2)$
$\frac{2}{5}$	$\frac{1}{8}$	$E_{12}$	$A_7$	$V(x_1x_2^3 + x_0x_3^3, x_1^2 - x_0x_2)$

TABLE 6. Summary VGIT

not have other bad geometric features – see Shah [Sha81, Theorem 4.8]) is  $t$ -stable for  $t < \frac{2}{5}$ . Then, at  $t = \frac{2}{5}$  becomes semistable with minimal orbit as given in Table 6.

We close by a general heuristic which gives a higher level explanation for why our predictions on the GIT behavior are reasonable:

*Remark 6.37* (Heuristic). One can view the change of stability as  $t$  varies from 0 to  $\frac{1}{2}$  as an interpolation from Shah’s GIT stability for  $(4, 4)$  curves  $C$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  to the stability imposed by being the ramification curve  $C$  for a  $K3$  (or an slc degeneration of it). Specifically, the ramification curve  $C$  for a  $K3$  will only have ADE singularities, and in the boundary cases simple elliptic, cusps, and some allowable non-isolated singularities (so that the double cover is slc). Thus, we expect that all the Type IV singularities from **Proposition 3.12** will be removed as  $t$  increases, and of course the worse singularities (e.g. the conic with multiplicity 4) before the milder one (e.g.  $E_{12}$ ; N.B. away from the equisingular stratum,  $E_{12}$  only deforms to simple elliptic, cusp, and ADE singularities). Of course, as we remove the bad singularities, there is a price to pay: we need to allow the quadric to be more singular, and the interactions of the curve  $C$  with the singularities of the quadric to become worse. As discussed, at  $t = \frac{1}{6}$  we allow the quadric to become a cone, and at the terminal value  $t = \frac{1}{2}$  we allow the quadric to become reducible (corresponding to the fact that a  $K3$  can degenerate to a reducible surface). The intermediary values between  $(\frac{1}{6}, \frac{1}{2})$  correspond to allowing the curve  $C$  worse and worse singularities at the vertex of quadric cone, which in turn is related by **Proposition B.1** to the  $Z_h^k$  loci. In conclusion, as  $t$  increases we will see  $K3$  surfaces deeper and deeper in the hyperelliptic locus (as measured of being in the  $k$ -fold intersection of the hyperelliptic divisor), while removing the non-slc singularities.

#### APPENDIX A. THE BAILY-BOREL COMPACTIFICATION FOR THE $D$ TOWER

Any locally symmetric variety  $\Gamma \backslash \mathcal{D}$  has a canonical projective compactification, the Baily-Borel compactification  $(\Gamma \backslash \mathcal{D})^* = \text{Proj} R(\Gamma \backslash \mathcal{D}, \mathcal{L})$ , where  $\mathcal{L}$  is the automorphic (or Hodge) line bundle. In the case of Type IV domains  $\mathcal{D}$ , the boundary consists of a union of modular curves, the *Type-II* boundary components, and some isolated points, the *Type-III* boundary components, which are in the closure of these modular curves. In the present section we will describe the boundary components of  $\mathcal{F}(N)^*$  for  $N \leq 20$ . The Baily-Borel boundary components for  $N = 19$  (quartic  $K3$ s) and  $N = 20$  (EPW sextics modulo duality) have been described by Scattone [Sca87] and



Camere [Cam15]. We extend their analysis to cover the lower dimensional cases (this is necessary for our inductive study). The number of boundary components are listed in Table 7. Within the

TABLE 7. Number of boundary components of  $\mathcal{F}(N)$

N	$\leq 9$	10	11–13	14	15–16	17	18	19	20
Type II	1	2	2	4	3	5	8	9	13
Type III	1	2	1	1	1	1	2	1	1

section we will provide a more detailed description of the boundary components, see **Proposition A.1** and **Theorem A.12**.

**A.1. Type II and Type III boundary components in general.** Let  $\Lambda$  be a lattice of signature  $(2, m)$ , and  $\Gamma < O^+(\Lambda)$  be a finite-index subgroup. For  $k \in \{1, 2\}$ , we let  $\mathcal{S}_k(\Lambda)$  be the set of rank- $k$  saturated isotropic subgroups of  $\Lambda$ . The set of Type-III boundary components of  $\mathcal{F}_\Lambda(\Gamma)$  is in one-to-one correspondence with  $\Gamma \backslash \mathcal{S}_1(\Lambda)$ , and the set of Type-II boundary components of  $\mathcal{F}_\Lambda(\Gamma)$  is in one-to-one correspondence with  $\Gamma \backslash \mathcal{S}_2(\Lambda)$ . Thus our task will be to enumerate  $\Gamma_{\xi_N}$ -orbits of elements of  $\mathcal{S}_k(\Lambda_N)$ , for  $k \in \{1, 2\}$ .

**A.2. Type III boundary components.**

**Proposition A.1.** *Let  $3 \leq N$ .*

- (1) *If  $N \not\equiv 2 \pmod{8}$ , there is one Type-III boundary components of  $\mathcal{F}(N)$ .*
- (2) *If  $N \equiv 2 \pmod{8}$ , there are two Type-III boundary components of  $\mathcal{F}(N)$ , one corresponds to the  $\Gamma_{\xi_N}$ -orbit in  $\mathcal{S}_1(\Lambda)$  whose elements are those  $L$  such that  $(L, \Lambda) = \mathbb{Z}$  and the other to the  $\Gamma_{\xi_N}$ -orbit whose elements are those  $L$  such that  $(L, \Lambda) = 2\mathbb{Z}$ .*

**Definition A.2.** Let  $3 \leq N$ . A Type III boundary component of  $\mathcal{F}(N)$  is of *class a* if it corresponds to the  $\Gamma_{\xi_N}$ -orbit of  $L \in \mathcal{S}_1(\Lambda)$  such that  $(L, \Lambda) = \mathbb{Z}$ , and it is of *class b* if it corresponds to the  $\Gamma_{\xi_N}$ -orbit of  $L \in \mathcal{S}_1(\Lambda)$  such that  $(L, \Lambda) = 2\mathbb{Z}$ .

*Remark A.3.* We have an injection  $f_{N+1}: \mathcal{F}(N) \hookrightarrow \mathcal{F}(N+1)$  with image  $H_h(N+1)$ . The map  $f_{N+1}$  extends to a finite map

$$(A.1) \quad f_{N+1}^*: \mathcal{F}(N)^* \longrightarrow \mathcal{F}(N+1)^*,$$

mapping Type II (Type III) boundary components to Type II (Type III) boundary components. Since the number of Type III boundary components is 1 if  $N \not\equiv 2 \pmod{8}$ , and 2 if  $N \equiv 2 \pmod{8}$ , it follows that if  $N \equiv 2 \pmod{8}$  then  $f_{N+1}^*$  is not injective, and  $H_h(N+1)^*$  (the closure of  $H_h(N+1)$  in  $\mathcal{F}(N+1)^*$ ) is not normal. In fact,  $\mathcal{F}(N)^* \longrightarrow H_h(N+1)^*$  is the normalization map (since it is birational, finite, with normal source) and the fiber over the unique Type III boundary point (of class *a*) consists of the two Type III boundary components, one of class *a*, the other of class *b*.

**A.3. Type II boundary components.** Let  $J \in \mathcal{S}_2(\Lambda_N)$ , i.e.  $J$  is a rank-2 saturated isotropic subgroup of  $\Lambda_N$ . The quadratic form on  $\Lambda_N$  induces a (negative-definite) quadratic form  $q_J$  on  $J^\perp/J$ . If  $J_1, J_2 \in \mathcal{S}_2(\Lambda_N)$  are  $O(\Lambda_N)$ -equivalent, then the corresponding lattices  $(J_r^\perp/J_r, q_{J_r})$  are isomorphic. Since  $\Gamma_{\xi_N} < O(\Lambda_N)$ , we have a well-defined map

$$(A.2) \quad \begin{array}{ccc} \Gamma_{\xi_N} \backslash \mathcal{S}_2(\Lambda_N) & \xrightarrow{\alpha_N} & \{\text{negative definite lattices of rank } (N-2)\} / \text{isom} \\ \Gamma_{\xi_N}\text{-orbit of } J & \mapsto & \text{isom. class of } J^\perp/J \end{array}$$

We will describe  $\Gamma_{\xi_N} \backslash \mathcal{S}_2(\Lambda_N)$  (i.e. the set of boundary components of Type II) by describing the image of  $\alpha_N$ , and by analyzing the fibers of  $\alpha_N$ .

**Proposition A.4.** *Let  $3 \leq N$ , and let  $J \subset \Lambda_N$  be a rank-2 saturated isotropic subgroup. Then one of the following holds:*

- (1) *The map  $\Lambda_N \rightarrow \text{Hom}(J, \mathbb{Z})$  defined by the quadratic form is surjective, and there exist a sublattice  $H \subset \Lambda_N$  isomorphic to  $U \oplus U$ , containing  $J$ , and a decomposition  $\Lambda_N = H \oplus L$ . Moreover  $J^\perp/J$  is isomorphic to  $L$ , a lattice in the genus of  $D_{N-2}$  (i.e. of rank  $(N-2)$ , even, negative definite, with  $(A_L, q_L) \cong (A_{D_{N-2}}, q_{D_{N-2}})$ ).*
- (2) *The map  $\Lambda_N \rightarrow \text{Hom}(J, \mathbb{Z})$  defined by the quadratic form has image a subgroup of index 2, and there exist a sublattice  $H \subset \Lambda_N$  isomorphic to  $U \oplus U(2)$ , containing  $J$ , and a decomposition  $\Lambda_N = H \oplus L$ . Moreover  $J^\perp/J$  is isomorphic to  $L$ , a rank  $(N-2)$  even negative definite unimodular lattice.*

*If  $N \not\equiv 2 \pmod{8}$  then Item (2) does not occur.*

**Corollary A.5.** *Let  $3 \leq N$ . The image of  $\alpha_N$  is equal to the set of isomorphism classes of*

- (1) *lattices in the genus of  $D_{N-2}$  (i.e. rank  $(N-2)$  even negative definite lattices  $L$  with  $(A_L, q_L) \cong (A_{D_{N-2}}, q_{D_{N-2}})$ ), if  $N \not\equiv 2 \pmod{8}$ ,*
- (2) *lattices which are in the genus of  $D_{N-2}$ , or are even, negative definite, unimodular, of rank  $(N-2)$ , if  $N \equiv 2 \pmod{8}$ .*

Now we restrict to the case  $3 \leq N \leq 20$ . In order to describe the image of  $\alpha_N$  we must classify the lattices appearing in **Corollary A.5** for  $N$  in the chosen range. As is well-known, there is only one even, negative definite, unimodular lattice of rank 8 up to isomorphism, namely  $E_8$ , and there are two isomorphism classes of even, negative definite, unimodular lattices of rank 16, namely  $E_8 \oplus E_8$ , and the unique (up to isomorphism) unimodular overlattice of  $D_{16}$ , call it  $D_{16}^+$ .

Before classifying lattices in the genus of  $D_n$  for  $1 \leq n \leq 18$ , we introduce a piece of notation. Given a negative definite lattice  $L$ , the *root lattice* of  $L$  is the sublattice  $R(L) \subset L$  generated (over  $\mathbb{Z}$ ) by roots of  $L$ , i.e. vectors  $v \in L$  such that  $v^2 = -2$ .

**Theorem A.6.** *Let  $1 \leq n \leq 18$ . The isomorphism class of a lattice  $M$  in the genus of  $D_n$  is determined by the isomorphism class of its root sublattice  $R(M)$ . The root sublattices that one gets are the following:*

- (1)  $n = 1$ :  $\emptyset$ .
- (2)  $2 \leq n \leq 8$ :  $D_n$ .
- (3)  $n = 9$ :  $D_9, E_8$ .
- (4)  $10 \leq n \leq 12$ :  $D_n, D_{n-8} \oplus E_8$ .
- (5)  $n = 13$ :  $D_{13}, D_5 \oplus E_8, D_{12}$ .
- (6)  $n = 14$ :  $D_{14}, D_6 \oplus E_8, D_{12} \oplus D_2$ .
- (7)  $n = 15$ :  $D_{15}, D_7 \oplus E_8, D_{12} \oplus D_3, A_{15}, (E_7)^2$ .
- (8)  $n = 16$ :  $D_{16}, D_8 \oplus E_8, D_{12} \oplus D_4, D_2 \oplus (E_7)^2, (D_8)^2, A_{15}$ .
- (9)  $n = 17$ :  $D_{17}, D_9 \oplus E_8, D_{12} \oplus D_5, D_3 \oplus (E_7)^2, A_{15} \oplus D_2, A_{11} \oplus E_6, (D_8)^2, D_{16}, (E_8)^2$ .
- (10)  $n = 18$ :  $D_{18}, D_{10} \oplus E_8, D_{12} \oplus D_6, D_4 \oplus (E_7)^2, A_{15} \oplus D_3, (D_8)^2 \oplus D_2, D_{16} \oplus D_2, D_2 \oplus (E_8)^2, A_{11} \oplus E_6, (D_6)^3, (A_9)^2, D_{10} \oplus E_7 \oplus A_1, A_{17} \oplus A_1$ .

*Moreover, if  $R(M)$  has rank strictly smaller than  $M$  (and hence  $\text{rk } R(M) = \text{rk } M - 1$ , by the above list), then  $M$  is an overlattice of  $R(M) \oplus D_1$ .*

**Remark A.7.** The most involved cases, namely  $n = 17$  and  $18$ , were previously studied by Scattone [Sca87, §6.3] and Camere [Cam15, Prop. 6.21] (based on Nishiyama [Nis96, Thm. 3.1]). The main method (occurring already in [Sca87]) for the proof of theorem is to study the embeddings of the  $D_k$  into the unimodular rank 24 lattices (the Niemeier lattices); the essential information is contained in Table 8 below.

$R(\text{Niemeier})$	Max Sat. $D_k$	Non-Sat. $D_k$	Min $N$
$D_{24}$	$D_{23}$	$D_{24}$	3
$\underline{D_{16}} \oplus E_8$	$D_{15}$	$D_{16}$	10
$(D_{12})^2$	$D_{12}$	–	14
$\underline{D_{10}} \oplus (E_7)^2$	$D_{10}$	–	16
$A_{15} \oplus D_9$	$D_9$	–	17
$(D_8)^3$	$D_8$	–	18
$D_{16} \oplus \underline{E_8}$	$D_7$	$D_8$	19
$(E_8)^3$	$D_7$	$D_8$	19
$A_{11} \oplus D_7 \oplus E_6$	$D_7$	–	19
$(D_6)^4$	$D_6$	–	20
$D_6 \oplus (A_9)^2$	$D_6$	–	20
$D_{10} \oplus \underline{(E_7)^2}$	$D_6$	–	20
$A_{17} \oplus E_7$	$D_6$	–	20

TABLE 8. Embeddings of  $D_k$  ( $k \geq 6$ ) into Niemeier lattices

*Remark A.8.* Many of the lattices in the genus of  $D_n$  listed above are obtained as an index 2 overlattice  $M$  of  $D_a \oplus D_b$  (with  $a + b = n$ ). We recall that such an  $M$  is determined by an order 2 isotropic subgroup  $H$  of  $A_{D_a} \oplus A_{D_b}$  (such that  $H^\perp/H \cong A_{D_n}$ ). Following the notation of [LO16] (see especially Subsection 1.1 of loc. cit.), we can take  $\alpha_a = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ ,  $\xi_a = (1, 0, \dots, 0)$ , and  $\beta_a = (-\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$  to be the non-zero elements of  $A_{D_a} = (D_a)^*/D_a$  (and similarly for  $D_b$ ). The choice  $H = \langle (\xi_a, \xi_b) \rangle \subset A_{D_a} \oplus A_{D_b}$  is always possible and leads to  $D_{n(=a+b)}$ . For special choices of  $a, b$ , there are additional choices for  $H$ , leading to different lattices in the genus of  $D_n$ . For instance, if  $a = b = 8$ , we can take  $H = \langle (\alpha_a, \alpha_b) \rangle \cong \mathbb{Z}/2$  which corresponds to adjoining to  $D_8 \oplus D_8$  the norm  $-4$  vector  $\frac{1}{2}((1, \dots, 1), (1, \dots, 1))$ . Similarly, if  $a = 12$ ,  $b \geq 1$ , we can take  $H = \langle (\alpha_{12}, \xi_b) \rangle$  which corresponds to adjoining to  $D_{12} \oplus D_b$  the norm  $-4$  vector  $\frac{1}{2}((1, \dots, 1), (2, 0, \dots, 0))$ .

In describing the fibers of  $\alpha_N$ , we will appeal to the following observation.

*Remark A.9.* Let  $X$  be a set, and  $G$  a group acting (on the left) on  $X$ . Let  $H < G$  be a finite-index subgroup, and  $\pi: H \backslash X \rightarrow G \backslash X$  the natural map. Then the fiber of  $\pi$  over the  $G$ -orbit  $[x]_G$  has cardinality given by

$$|\pi^{-1}([x]_G)| = \frac{[G : H]}{[\text{Stab}_G(x) : \text{Stab}_H(x)]},$$

where  $\text{Stab}_G(x)$  and  $\text{Stab}_H(x)$  are the stabilizers of  $x$  in  $G$  and  $H$  respectively.

We will also need the following elementary result.

**Lemma A.10.** (1) *The group  $O(U \oplus U)$  acts transitively on  $\mathcal{I}_2(U \oplus U)$ . Moreover, given  $J \in \mathcal{I}_2(U \oplus U)$ , there exists  $\varphi \in (O(U \oplus U) \setminus O^+(U \oplus U))$  which maps  $J$  to itself.*

- (2) The group  $O(U \oplus U(2))$  acts transitively on the set of  $J \in \mathcal{J}_2(U \oplus U(2))$  such that the map  $(U \oplus U(2)) \rightarrow \text{Hom}(J, \mathbb{Z})$  is not surjective. Moreover, given  $J$  as above, there exists  $\varphi \in (O(U \oplus U) \setminus O^+(U \oplus U))$  which maps  $J$  to itself.

**Proposition A.11.** *Let  $3 \leq N$ . If  $N \not\equiv 6 \pmod{8}$ , the map  $\alpha_N$  in (A.2) is injective. The map  $\alpha_6$  is injective as well. The fiber of  $\alpha_{14}$  over the isomorphism class of  $D_{12}$  has cardinality 3, while the fiber over the isomorphism class of  $D_4 \oplus E_8$  is a singleton.*

We conclude:

**Theorem A.12.** *Let  $3 \leq N \leq 20$ . The number of Type II components of  $\mathcal{F}(N)$  is given in **Table 7**. More precisely,*

- (1) *For  $N \not\equiv 2 \pmod{8}$ , Type II components are uniquely labeled by the lattices in the genus of  $D_{N-2}$  (as listed in **Theorem A.6**), with the exception of  $N = 14$ , in this case there are three Type II components labeled by  $D_{12}$ . The Type II components are pairwise disjoint, but their closures all meet in a single point, corresponding to the unique Type III boundary point (cf. **Proposition A.1**).*
- (2) *If  $N \equiv 2 \pmod{8}$ , there are two kinds of Type II components, those mapped by  $\alpha_N$  to the isomorphism class of a lattice in the genus of  $D_{N-2}$ , and those which are mapped by  $\alpha_N$  to the isomorphism class of an even negative definite unimodular lattice of rank  $N - 2$ ; in both cases the isomorphism class of the corresponding lattice uniquely determined the boundary component. The Type II components are pairwise disjoint, but the closure of each component of the first kind contains the Type III point of class  $a$  (see **Definition A.2**) and not the Type III component of class  $b$ , while the closure of each Type II component of the second kind contains both Type III boundary points.*

Diagram (A.3) illustrates the structure of the Baily–Borel boundary for  $N \in \{9, 10, 11\}$ . Type II and Type III boundary components are represented by  $\circ$  and  $\bullet$  respectively. We recall that a Type III component in the Baily–Borel compactification is just a point, while Type II components are modular curves  $G \backslash \mathfrak{h}$ , where  $\mathfrak{h}$  is the Siegel upper half-space and  $G < \text{SL}(2, \mathbb{Q})$  is a subgroup commensurable to  $\text{SL}(2, \mathbb{Z})$  (typically  $G = \text{SL}(2, \mathbb{Z})$ , e.g. this is the case for all Type II components for quartic  $K3$ s ( $N = 19$ ) cf. Scattone [Sca87, Fig. 5.5.7, on p. 70]). The incidence between Type II and Type III (meaning that a Type III component is contained in the closure of a Type II component) is illustrated by a line.

$$\begin{aligned}
 \text{(A.3)} \quad N = 9 : & \quad \bullet^{III_a} \text{ --- } \circ^{II(D_7)} \\
 N = 10 : & \quad \bullet^{III_b} \text{ --- } \circ^{II(E_8)} \text{ --- } \bullet^{III_a} \text{ --- } \circ^{II(D_8)} \\
 N = 11 : & \quad \circ^{II(E_8 \oplus D_1)} \text{ --- } \bullet^{III_a} \text{ --- } \circ^{II(D_9)}
 \end{aligned}$$

The appendices  $a$  and  $b$  for Type III components refer to the notation introduced in **Definition A.2**.

*Remark A.13.* Let  $f_{N+1}^* : \mathcal{F}(N)^* \rightarrow \mathcal{F}(N+1)^*$  be the map in (A.1). Then  $f_{N+1}^*$  maps a Type II boundary component of  $\mathcal{F}(N)^*$  to a Type II boundary component of  $\mathcal{F}(N+1)^*$ , and the inverse image by  $f_{N+1}^*$  of a Type II boundary component of  $\mathcal{F}(N+1)^*$  is a union of Type II boundary components of  $\mathcal{F}(N)^*$ . The corresponding map between Type II boundary components is related to inclusion relations among the lattices listed in **Theorem A.6** for  $n \in \{N-2, N-1\}$  (we must include also the lattice  $E_8$  if  $N \in \{9, 10\}$ , and the lattices  $E_8 \oplus E_8$ ,  $D_{16}^+$  if  $N \in \{18, 19\}$ ).

APPENDIX B.  $\mathcal{F}(18-k)$  AND SPECIAL HYPERELLIPTIC QUARTIC  $K3$  SURFACES

Below is the main result of the present subsection.

**Proposition B.1.** *Let  $x \in \mathcal{F}(18)$ , and let  $(X, L)$  be a hyperelliptic quartic  $K3$  surface whose period point is  $x$  ( $(X, L)$  is unique up to isomorphism by Global Torelli). Let  $Q := \varphi_L(X)$  be the image quadric in  $|L|^\vee \cong \mathbb{P}^3$ , and let  $B \in |\omega_Q^{-2}|$  be the branch divisor of  $X \rightarrow Q$ . The following hold:*

- (1) *Let  $1 \leq k \leq 15$ . Then  $x \in \text{Im } f_{18-k,18}$  if and only if  $Q$  is a quadric cone, and  $B$  has an  $A_m$ -singularity at the vertex of  $Q$ , where  $m \geq (k-1)$ , and, if  $k = 8$ ,  $B$  does not contain a line.*
- (2)  *$x \in \text{Im}(f_{11,18} \circ l_{11})$  if and only if  $B$  contains a line, and has an  $A_7$ -singularity at the vertex of  $Q$ .*
- (3)  *$x \in \text{Im}(f_{12,18} \circ m_{12})$  if and only if  $B$  contains a line, and has an  $A_m$ -singularity at the vertex of  $Q$ , where  $m \geq 5$ .*

We will prove **Proposition B.1** at the end of the present subsection. The remark below serves the purpose of introducing notation for hyperelliptic surfaces which are double covers of a quadric cone.

*Remark B.2.* Let  $Q \subset \mathbb{P}^3$  be a quadric cone, with vertex  $v$ . Let  $B \in |\omega_Q^{-2}|$ , and suppose that  $B$  has simple singularities. Let  $\varphi: X \rightarrow Q$  be the double cover ramified over  $B$ , and let  $L := \varphi^* \mathcal{O}_Q(1)$ . Then  $(X, L)$  is a hyperelliptic quartic  $K3$  surface, and the image of  $\varphi_L: X \rightarrow |L|^\vee$  is identified with  $Q$ . We let  $\tilde{X} \rightarrow X$  be the minimal desingularization of  $X$ . Let  $\mu_0: Q_0 \rightarrow Q$  be the blow up of the vertex  $v$ ; thus  $Q_0 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{F}_2$ . Let  $A, F \subset Q_0$  be the negative section and a fiber of the  $\mathbb{P}^1$ -fibration  $Q_0 \rightarrow \mathbb{P}^1$ , respectively. Let  $B_0 := \mu_0^* B \in |4A + 8F|$ , and let  $\varphi_0: X_0 \rightarrow Q_0$  be the double cover ramified over  $B_0$ . Then  $\tilde{X}$  is the minimal desingularization of  $X_0$ , and hence  $X_0$  is a  $K3$  surface as well. It follows that  $B_0$  has simple singularities. Conversely, if  $B_0$  has simple singularities, then  $X$  is a  $K3$  surface. We let  $\tilde{\varphi}_0: \tilde{X} \rightarrow Q_0$  be the composition  $\tilde{X} \rightarrow X_0 \rightarrow Q_0$ .

**Proposition B.3.** *Let  $(X, L)$  be a hyperelliptic quartic  $K3$ . Then  $\Pi(X, L) \in H_h(18)$  if and only if the image quadric  $\varphi_L(X) \subset |L|^\vee$  is a cone.*

*Proof.* Let  $Q := \varphi_L(X)$ . Suppose that  $Q$  is a quadric cone, and let us adopt the notation of **Remark B.2**. Assume that the vertex  $v$  of  $Q$  is not in the support of the branch divisor  $B \in |\omega_Q^{-2}|$ . Let  $\tilde{\varphi}_0: \tilde{X} \rightarrow Q_0$  be the degree-2 map of **Remark B.2**. Since  $A := \mu_0^{-1}(v)$ , the support of the branch divisor  $B_0 = \mu_0^* B$  is disjoint from  $A$ , and hence  $\tilde{\varphi}_0^* A = A_1 + A_2$ , where  $A_1, A_2 \subset \tilde{X}$  are disjoint smooth rational curves. Thus  $A_1 \cdot \tilde{\varphi}_0^* F = A_2 \cdot \tilde{\varphi}_0^* F = 1$ ; it follows that  $[A_1 - A_2] \in \tilde{\varphi}_0^* H^2(Q_0; \mathbb{Z})^\perp$  (here  $[A_1 - A_2] \in H^2(\tilde{X}; \mathbb{Z})$  is the Poincaré dual of the divisor  $A_1 - A_2$ ). Now let  $\psi: H^2(Q_0; \mathbb{Z})^\perp \xrightarrow{\sim} \Lambda_{18}$  be an isomorphism. Then  $\Pi(X, L) = \psi_{\mathbb{C}} H^{2,0}(\tilde{X})$ . Since  $\psi_{\mathbb{C}} H^{2,0}(\tilde{X}) \in \psi([A_1 - A_2])^\perp$ , it will suffice to prove that  $\psi([A_1 - A_2])$  is a hyperelliptic vector. First,  $\psi([A_1 - A_2])^2 = [A_1 - A_2]^2 = -4$ , secondly  $\psi([A_1 - A_2])$  has divisibility 2 because  $[A_1 - A_2] + \tilde{\varphi}_0^* A = 2A_1$  (we recall that a vector in  $\Lambda_{18}$  of square  $-4$  has divisibility 1 or 2). This proves that if  $Q$  is a quadric cone, and the branch divisor  $B \in |\omega_Q^{-2}|$  does not contain the vertex of the cone, then  $\Pi(X, L) \in H_h(18)$ . Since  $H_h(18)$  is closed in  $\mathcal{F}(18)$ , the same is true whenever  $Q$  is a quadric cone. By irreducibility of  $H_h(18)$ , and a dimension count, the proposition follows.  $\square$

*Remark B.4.* Retain the notation of **Remark B.2**. Suppose that the branch divisor  $B$  contains the vertex  $v$ , and that the double cover  $\varphi: X \rightarrow Q$  ramified over  $B$  is a  $K3$ . Then  $B_0 = A + C$ , where  $C \in |3A + 8F|$ , and  $B_0$  has simple singularities. In particular  $C$  does not contain  $A$ . Now notice that  $C \cdot A = 2$ . Thus, either  $C$  intersects transversely  $A$  in 2 points, or it intersects  $A$  in a single point  $p$ . Assume that the latter holds. Since  $B_0 = A + C$  has simple singularities, it follows



that  $C$  has an  $A_n$ -singularity at  $p$  (if  $n = 0$ , this means that  $C$  is smooth at  $p$ , and simply tangent to  $A$ ), and simple singularities elsewhere. Conversely, if  $C$  is as described above, then  $X_0$  is a  $K3$ , and hence so is  $X$ . For  $k \geq 2$  the following are equivalent:

- (1)  $B$  has an  $A_{k-1}$ -singularity at  $v$ .
- (2)  $C$  has an  $A_{k-3}$ -singularity at the point(s) of intersection with  $A$ .
- (3)  $B_0$  has a  $D_k$ -singularity at the point(s) of intersection with  $A$ .

(If  $k = 2$ , then Item (2) is to be interpreted as saying that  $C$  intersects  $A$  transversely in two points, and Item (3) accordingly.)

**Definition B.5.** Let  $Q \subset \mathbb{P}^3$  be a quadric cone, with vertex  $v$ . Let  $|\omega_Q^{-2}|_{K3} \subset |\omega_Q^{-2}|$  be the (open) subset of  $B$  with simple singularities (i.e. such that the double cover  $X \rightarrow Q$  ramified over  $B$  is a  $K3$ ), and let  $S_k(Q) \subset |\omega_Q^{-2}|_{K3}$  be the set of divisors with an  $A_{k-1}$ -singularity at  $v$ .

Notice that  $S_k(Q)$  is locally closed in  $|\omega_Q^{-2}|_{K3}$ . By **Remark B.4**,

$$(B.1) \quad |\omega_Q^{-2}|_{K3} = \bigcup_{k=1}^{\infty} S_k(Q).$$

**Proposition B.6.** Let  $Q \subset \mathbb{P}^3$  be a quadric cone, with vertex  $v$ . Let  $B \in S_k(Q)$ , with  $k \geq 1$ . Let  $\varphi: X \rightarrow Q$  be the double cover ramified over  $B$ , and  $L := \varphi^* \mathcal{O}_Q(1)$ . Then

$$\Pi(X, L) \in \begin{cases} \text{Im } f_{18-k, 18} & \text{if } k \neq 8, \\ \text{Im } f_{10, 18} \cup \text{Im}(f_{11, 18} \circ l_{11}) & \text{if } k = 8. \end{cases}$$

*Proof.* We may assume that  $k \geq 2$ , because for  $k = 1$  the statement follows from **Proposition B.3**. Let  $R \subset H^{1,1}(\tilde{X}; \mathbb{Z})$  be the span of the classes of curves which are mapped to a point of  $A \cap C$  (notation as in **Remark B.2**) by  $\tilde{\varphi}_0$ . Then the following hold:

- (1)  $R \subset \tilde{\varphi}_0^* H^2(Q_0; \mathbb{Z})^\perp$ ,
- (2)  $R \cong D_k$ ,

Item (1) is clear. If  $k = 2$ , then  $C$  intersects  $A$  at two points, and the intersection is transverse; thus  $R \cong A_1^2 \cong D_2$ . If  $k \geq 3$  then  $C$  intersects  $A$  at a single point  $p$ , which is a  $D_k$ -singularity of  $B_0$  (see **Remark B.4**), and Item (2) follows. Let  $\psi: \tilde{\varphi}_0^* H^2(Q_0; \mathbb{Z})^\perp \xrightarrow{\sim} \Lambda_{18}$  be an isomorphism such that  $\psi_{\mathbb{C}} H^{2,0}(\tilde{X}) \in \mathcal{D}_{\Lambda_{18}}^+$ . Then  $\Pi(X, L) \subset \psi(R)^\perp$ ; thus, by [LO16, Proposition 1.7.2] it will suffice to prove that  $\psi(R)$  contains  $k$  pairwise orthogonal hyperelliptic vectors. A straightforward computation shows that

$$\tilde{\varphi}_0^* A = 2\Gamma_0 + 2\Gamma_1 + \dots + 2\Gamma_{k-2} + \Gamma_{k-1} + \Gamma_k,$$

where  $\tilde{\varphi}_0(\Gamma_0) = A$ ,  $\tilde{\varphi}_0^{-1}(A \cap C) = \Gamma_1 \cup \dots \cup \Gamma_k$ , and the rational curves  $\Gamma_0, \Gamma_1, \dots, \Gamma_k$  form a  $D_{k+1}$ -Dyinkin diagram

$$(B.2) \quad \begin{array}{ccccccc} & & & & & \Gamma_{k-1} & \\ & & & & & \swarrow & \\ & & & & & \Gamma_{k-2} & \\ & & & & & \searrow & \\ & & & & & \Gamma_k & \\ \Gamma_0 & \text{---} & \Gamma_1 & \cdots & \Gamma_l & \cdots & \Gamma_{k-2} \end{array}$$

Thus  $R$  is spanned by the cohomology classes  $[\Gamma_1], \dots, [\Gamma_k]$ . The  $k$  cohomology classes

$$(B.3) \quad [\Gamma_k] - [\Gamma_{k-1}], [\Gamma_k] + [\Gamma_{k-1}], [\Gamma_k] + [\Gamma_{k-1}] + 2[\Gamma_{k-2}] + \dots + 2[\Gamma_i], \quad 1 \leq i \leq k-2,$$

are pairwise orthogonal, and of square  $-4$ . In order to finish the proof it suffices to show that each of the classes in (B.3) has divisibility 2 in  $\tilde{\varphi}_0^* H^2(Q_0; \mathbb{Z})^\perp$ . Let  $1 \leq i \leq k-1$ , and  $z \in \tilde{\varphi}_0^* H^2(Q_0; \mathbb{Z})^\perp$ ; then

$$0 = z \cdot \tilde{\varphi}_0^* A = z \cdot (2\Gamma_0 + 2\Gamma_1 + \dots + 2\Gamma_{k-2} + \Gamma_{k-1} + \Gamma_k) \equiv z \cdot (2\Gamma_i + 2\Gamma_{i+1} + \dots + 2\Gamma_{k-2} + \Gamma_{k-1} + \Gamma_k) \pmod{2}$$

(If  $i = k - 1$ , the expression in parentheses is to be understood as  $(\Gamma_{k-1} + \Gamma_k)$ .) This proves that all of the classes listed in (B.3) have divisibility a multiple of 2. On the other hand

$$\Gamma_k \cdot (\Gamma_k \pm \Gamma_{k-1}) = -2, \quad \Gamma_i \cdot (\Gamma_k + \Gamma_{k-1} + 2\Gamma_{k-2} + \dots + 2\Gamma_i) = -2, \quad 1 \leq i \leq k-2,$$

and hence their divisibility is equal to 2.  $\square$

**Proposition B.7.** *Let  $Q \subset \mathbb{P}^3$  be a quadric cone, with vertex  $v$ . Then  $S_k(Q)$  is not empty for  $1 \leq k \leq 16$ , and each of its irreducible components has codimension at most  $(k-1)$  in  $|\omega_Q^{-2}|_{K3}$ . There exist irreducible components  $Z_k(Q)$  of  $S_k(Q)$ , for  $1 \leq k \leq 16$ , such that  $Z_k(Q) \subset \overline{Z_{k-1}(Q)}$  for  $2 \leq k \leq 16$ .*

*Proof.* First we prove that  $S_k(Q)$  is not empty, and that its codimension is as stated. If  $k = 1$ , the result is trivially true. Since  $S_2(Q)$  is an open dense subset of  $|\mathcal{I}_v \otimes \omega_Q^{-2}|$ , the statement of the proposition is true for  $k = 2$ . It is equally easy to check that the statement is true for  $k = 3$ , if we identify  $S_3(Q)$  with the set of  $C \in |3A + 8F|$  with simple singularities, which intersect  $A$  at a single point, and are smooth at that point, see **Remark B.4**. Thus we may assume that  $k \geq 4$ . By **Remark B.4** we may identify  $S_k(Q)$  with the subset  $T_k(Q) \subset |3A + 8F|$  parametrizing divisors  $C$  with simple singularities, and with an  $A_{k-3}$ -singularity at the unique point of intersection with  $A$ . Once we know that  $T_k(Q)$  is not empty, the codimension statement follows from general results about the versal deformation space of an  $A_{k-3}$ -singularity. Let us prove that  $T_k(Q)$  is not empty. There exist  $C_1 \in |A + 3F|$ , and  $C_2 \in |2A + 5F|$  intersecting at a point  $p \in A$  (notice that  $A \cdot (A + 3F) = 1$ ) with multiplicity  $7 = (A + 3F) \cdot (2A + 5F)$ , and smooth at  $p$ . One argues that  $C_1, C_2$  exist via a simple cohomological argument. The following is an explicit example: let  $Q$  be the quadric cone  $V(z_1 z_3 - z_2^2)$ , and  $B = \text{div}(f)|_Q$ , where

$$f := z_0^3 z_3 - 2z_0^2 z_1 z_2 + z_0^2 z_3^2 - z_0 z_1^3 - z_1^3 z_3 + z_1 z_2 z_3^2.$$

This proves that  $T_{16}(Q)$  is not empty. Let  $\varphi: X \rightarrow Q$  be the double cover ramified over  $B$ , and let  $\tilde{X}$  be the minimal desingularization of  $X$ . Let  $\Gamma_0, \Gamma_1, \dots, \Gamma_{16} \subset \tilde{X}$  be the rational curves introduced in the proof of **Proposition B.6**. In the versal deformation of  $\tilde{X}$ , consider deformations that keep the classes  $\tilde{\varphi}_0^* F, [\Gamma_0 + \Gamma_1], [\Gamma_2], \dots, [\Gamma_k]$  of type  $(1, 1)$ . The generic such deformation is the desingularization of a double cover  $X' \rightarrow Q_0$  ramified over a divisor  $A + B'$ , where  $B' \in T_{15}(Q)$ . Iterating, one proves that  $T_k(Q)$  is not empty for  $1 \leq k \leq 16$ . This argument also produces irreducible components  $Z_k(Q)$  of  $S_k(Q)$  with the stated property.  $\square$

*Proof of Proposition B.1.* Since  $\text{Im } f_{18-k,18} \subset H_h(18)$  for  $1 \leq k$ , and  $\Pi(X, L) \in H_h(18) \cong \mathcal{F}(17)$  if and only if  $\varphi_L(X)$  is a quadric cone (by **Proposition B.3**), we may fix a quadric cone  $Q \subset \mathbb{P}^3$ , with vertex  $v$ , and consider only double covers  $X \rightarrow Q$  ramified over divisors  $B \in |\omega_Q^{-2}|_{K3}$ . Let

$$(B.4) \quad |\omega_Q^{-2}|_{K3} \xrightarrow{\Phi} \mathcal{F}(17)$$

be the map associating to  $B$  the period point  $\Pi(X, L)$ , where  $X \rightarrow Q$  is the double cover ramified over  $B$ . Then  $\Phi$  is a regular map of quasi-projective varieties, and by Global Torelli the fibers of  $\Phi$  are the orbits for the action of  $\text{Aut}(Q)$  on  $|\omega_Q^{-2}|_{K3}$ . The image by  $\Phi$  of an open subset of  $|\omega_Q^{-2}|_{K3}$  is open in  $\mathcal{F}(17)$ . It follows that the image by  $\Phi$  of a closed and  $\text{Aut}(Q)$ -invariant subset of  $|\omega_Q^{-2}|_{K3}$  is closed.

Let us prove Items (1) and (2). The subset  $\Phi(\overline{S(Q)_k}) \subset \mathcal{F}(18)$  is closed because  $\overline{S(Q)_k}$  is closed and  $\text{Aut}(Q)$ -invariant. Moreover

$$(B.5) \quad \Phi(\overline{S(Q)_k}) \subset \begin{cases} \text{Im } f_{18-k,18} & \text{if } k \neq 8, \\ \text{Im } f_{10,18} \cup \text{Im}(f_{11,18} \circ l_{11}) & \text{if } k = 8. \end{cases}$$

by **Proposition B.6**. On the other hand, by **Proposition B.7** every irreducible component of  $\overline{S(Q)}_k$  has codimension at most  $(k-1)$  in  $|\omega_Q^{-2}|_{K3}$ .

Now assume that  $k \neq 8$ . Since  $\text{Im } f_{18-k,18}$  is irreducible, closed, of codimension  $k$  in  $\mathcal{F}(18)$  (and hence of codimension  $k-1$  in  $H_h(18) \cong \mathcal{F}(17)$ ), it follows from (B.5) that  $\Phi(\overline{S(Q)}_k) = \text{Im } f_{18-k,18}$ .

Next, assume that  $k = 8$ . Let  $S_8^{\text{ex}}(Q) \subset S_8(Q)$  be the subset of divisors  $(L+D)$ , where  $L$  is a line on  $Q$ . Then  $S_8^{\text{ex}}(Q)$  is closed,  $\text{Aut}(Q)$ -invariant, and a straightforward parameter count shows that  $S_8^{\text{ex}}(Q)$  has codimension 7 in  $|\omega_Q^{-2}|$ . Recalling [LO16, Proposition 1.7.2], and arguing as above, it follows that  $\Phi(S_8^{\text{ex}}(Q))$  is an irreducible component of  $\text{Im } f_{10,18} \cup \text{Im}(f_{11,18} \circ l_{11})$ . On the other hand,  $S_8^{\text{ex}}(Q)$  is not the whole of  $S_8(Q)$ , because  $S_8(Q)$  is not closed, by **Proposition B.7**. It follows that there is a unique other irreducible component of  $S_8(Q)$  (uniquity follows from [LO16, Proposition 1.7.2], call it  $S_8^{\text{st}}(Q)$ , and that

$$\Phi(\overline{S_8^{\text{st}}(Q)}) = \text{Im } f_{10,18}, \quad \Phi(S_8^{\text{ex}}(Q)) = \text{Im}(f_{11,18} \circ l_{11}).$$

Now let us prove Item (3). Suppose that  $B \in S_6(Q)$  contains a line. Then, retaining the notation of **Remark B.2** and **Remark B.4**, we have  $C = C' + D$  with

$$C' \in |3A + 7F|, \quad D \in |F|.$$

Moreover there is a unique point in  $A \cap C' \cap D$ , call it  $p$ , and  $\text{mult}_p(C' \cdot D) = 2$  (because  $C$  has an  $A_3$  singularity at  $p$ , see **Remark B.4**). Since  $C' \cdot D = 3$ , there is a unique point in  $(C' \cap D) \setminus A$ , call it  $q$ , and  $\text{mult}_q(C' \cdot D) = 1$ . Now let  $(X, L)$  be the hyperelliptic quartic  $K3$  surface corresponding to  $B$ , i.e.  $\varphi: X \rightarrow Q$  is the double cover branched over  $B$  and  $L = \varphi^* \mathcal{O}_Q(1)$ . Then  $\tilde{\varphi}_0^{-1}(q)$  is a smooth rational curve  $N \subset \tilde{X}$  (notation as in **Remark B.2**). Next, let  $M \subset H^{1,1}(\tilde{X}; \mathbb{Z})$  be the subgroup spanned by  $\tilde{\varphi}_0^* H^2(Q_0; \mathbb{Z})$  and the classes of curves which are mapped to  $p$  by  $\tilde{\varphi}_0$  (this is the subspace denoted by  $R$  in the proof of **Proposition B.6**). Notice that  $c_1(\mathcal{O}_{\tilde{X}}(N)) \in M^\perp$ . We claim that

$$(B.6) \quad \text{div}_{M^\perp}(c_1(\mathcal{O}_{\tilde{X}}(N))) = 2.$$

In fact, letting  $\tilde{D} \subset \tilde{X}$  be the strict transform of  $D$ , and retaining the notation introduced in the proof of **Proposition B.6**,

$$(B.7) \quad \tilde{\varphi}_0^* D = N + 2\tilde{D} + \Gamma_1 + 2\Gamma_2 + 3\Gamma_3 + 4\Gamma_4 + 3\Gamma_5 + 2\Gamma_6.$$

Since  $\tilde{\varphi}_0^* D, \Gamma_1, \dots, \Gamma_5 \in M$ , Equation (B.6) follows at once from (B.7). Since  $c_1(\mathcal{O}_{\tilde{X}}(N)) \in M^\perp$ ,  $N \cdot N = -2$ , and  $\text{div}_{M^\perp}(c_1(\mathcal{O}_{\tilde{X}}(N))) = 2$ , it follows that  $\Pi(X, L) \in \text{Im}(f_{12,18} \circ m_{12})$ .

Now, let  $Z \subset S_6(Q)$  be the subset of  $B$  containing a line, and  $\overline{Z}$  its closure in  $S_6(Q)$ . We proved above that  $\Phi(Z) \subset \text{Im}(f_{12,18} \circ m_{12})$ , and since  $\text{Im}(f_{12,18} \circ m_{12})$  is closed, it follows that

$$(B.8) \quad \Phi(\overline{Z}) \subset \text{Im}(f_{12,18} \circ m_{12}).$$

On the other hand, an easy dimension count shows that  $\overline{Z}$  has codimension 6 in  $|\omega_Q^{-2}|_{K3}$ ; since  $\overline{Z}$  is  $\text{Aut}(Q)$ -invariant, it follows that  $\Phi(\overline{Z})$  is closed, of codimension 6 in  $\mathcal{F}(17)$ . Since  $\text{Im}(f_{12,18} \circ m_{12})$  is irreducible of codimension 6 in  $\mathcal{F}(17)$ , Item (3) of **Proposition B.1** follows from (B.8).  $\square$

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